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GROMOV–WITTEN INVARIANTS AND PSEUDO SYMPLECTIC CAPACITIES

BY

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ABSTRACT

We introduce the concept of pseudo symplectic capacities which is a mild generalization of that of symplectic capacities. As a generalization of the Hofer–Zehnder capacity we construct a Hofer–Zehnder type pseudo symplectic capacity and estimate it in terms of Gromov–Witten invariants. The (pseudo) symplectic capacities of Grassmannians and some product symplectic manifolds are computed. As applications we first derive some general nonsqueezing theorems that generalize and unite many previous versions, then prove the Weinstein conjecture for cotangent bundles over a large class of symplectic uniruled manifolds (including the uniruled manifolds in algebraic geometry) and also show that any closed symplectic submanifold of codimension two in any symplectic manifold has a small neighborhood whose Hofer–Zehnder capacity is less than a given positive number. Finally, we give two results on symplectic packings in Grassmannians and on Seshadri constants.

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1. Introduction and main results

Gromov–Witten invariants and symplectic capacities are two kinds of important symplectic invariants in symplectic geometry. Both have many important applications. In particular, they are related to the famous Weinstein conjecture and Hofer geometry (cf. [En, FrGiSchl, FrSchl, HZ2, HV2, LaMc1, LaMc2, LiuT, Lu1, Lu2, Lu3, Lu5, Lu7, Lu9, Mc2,Mc3, McSl, Po1,Po2, Po3, Schl, Schw, V1, V2,V3, V4, We2] etc.). For some problems, Gromov–Witten invariants are convenient and effective, but for other problems symplectic capacities are more powerful. In the study of different problems different symplectic capacities were defined. Examples of symplectic capacities are the Gromov width $\mathcal{W}_G$ ([Gr]), the Ekeland–Hofer capacity c_{EH} ([EH]), the Hofer–Zehnder capacity c_{HZ} ([HZ1]) and Hofer's displacement energy e ([H1]), the Floer–Hofer capacity c_{FH} ([He]) and Viterbo's generating function capacity c_V ([V3]). Only $\mathcal{W}_G$, c_{HZ} and e are defined for all symplectic manifolds. In [HZ1] an axiomatic definition of a symplectic capacity was given. The Gromov width $\mathcal{W}_G$ is the smallest symplectic capacity. The Hofer–Zehnder capacity is used in the study of many symplectic topology questions. The reader can refer to [HZ2, McSa1, V2] for more details. But to the author's knowledge the relations between Gromov–Witten invariants and symplectic capacities have not been explored explicitly in the literature. Gromov–Witten invariants are defined for closed symplectic manifolds ([FO, LiT, R, Sie]) and some non-closed symplectic manifolds (cf. [Lu4, Lu8]) and have been computed for many closed symplectic manifolds. However, it is difficult to compute c_{HZ} for a closed symplectic manifold. So far the only examples are closed surfaces, for which c_{HZ} is the area ([Sib]), and complex projective space $(\mathbb{C}P^n, \sigma_n)$ with the standard symplectic structure σ_n related to the

Fubini–Study metric: Hofer and Viterbo proved $c_{HZ}(\mathbb{C}P^n, \sigma_n) = \pi$ in [HV2]. Perhaps the invariance of Gromov–Witten invariants under deformations of the symplectic form is the main reason why it is easier to compute them than Hofer–Zehnder capacities. Unlike Gromov–Witten invariants, symplectic capacities do not depend on homology classes of the symplectic manifolds in question. We believe that this is a reason why they are difficult to compute or estimate, and it is based on this observation that we introduced the concept of pseudo symplectic capacities in the early version [Lu5] of this paper.

1.1 PSEUDO SYMPLECTIC CAPACITIES. In [HZ1] a map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a symplectic capacity if it satisfies the following properties:

(**monotonicity**) If there is a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ of codimension zero then $c(M_1, \omega_1) \le c(M_2, \omega_2)$;

(**conformality**) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$ for every $\lambda \in \mathbb{R} \setminus \{0\}$;

(**nontriviality**) $c(B^{2n}(1), \omega_0) = \pi = c(Z^{2n}(1), \omega_0)$.

Here $B^{2n}(1)$ and $Z^{2n}(1)$ are the closed unit ball and closed cylinder in the standard space $(\mathbb{R}^{2n}, \omega_0)$, i.e., for any $r > 0$,

$$B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} | \, |x|^2 + |y|^2 \le r^2\}$$

and

$$Z^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} | \, x_1^2 + y_1^2 \le r^2\}.$$

Note that the first property implies that c is a symplectic invariant.

Let $H_*(M; G)$ denote the singular homology of M with coefficient group G. For an integer $k \ge 1$ we denote by $\mathcal{C}(2n, k)$ the set of all tuples $(M, \omega; \alpha_1, \ldots, \alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold (M, ω) and nonzero homology classes $\alpha_i \in H_*(M; G)$, $i = 1, \ldots, k$. We denote by *pt* the homology class of a point.

Definition 1.1: A map $c^{(k)}$ from $\mathcal{C}(2n, k)$ to $[0, +\infty]$ is called a G_k-**pseudo symplectic capacity** if it satisfies the following conditions.

P1. **Pseudo monotonicity**: If there is a symplectic embedding ψ: $(M_1, \omega_1) \to (M_2, \omega_2)$ of codimension zero, then for any $\alpha_i \in H_*(M_1; G) \setminus \{0\}$, $i = 1, \ldots, k$,

$$c^{(k)}(M_1, \omega_1; \alpha_1, \ldots, \alpha_k) \le c^{(k)}(M_2, \omega_2; \psi_*(\alpha_1), \ldots, \psi_*(\alpha_k)).$$

P2. **Conformality:** $c^{(k)}(M, \lambda\omega; \alpha_1, \ldots, \alpha_k) = |\lambda| c^{(k)}(M, \omega; \alpha_1, \ldots, \alpha_k)$ for every $\lambda \in \mathbb{R} \setminus \{0\}$ and all homology classes $\alpha_i \in H_*(M; G) \setminus \{0\}$, $i = 1, \ldots, k$;

P3. **Nontriviality:** $c^{(k)}(B^{2n}(1),\omega_0;pt,\dots,pt)=\pi$
$$=c^{(k)}(Z^{2n}(1),\omega_0;pt,\dots,pt).$$

The pseudo monotonicity is the reason that a pseudo symplectic capacity in general fails to be a symplectic invariant. If $k>1$ then a G_{k-1}-pseudo symplectic capacity $c^{(k-1)}$ is naturally defined by

$$c^{(k-1)}(M,\omega;\alpha_1,\dots,\alpha_{k-1}):=c^{(k)}(M,\omega;pt,\alpha_1,\dots,\alpha_{k-1}),$$

and any $c^{(k)}$ induces a true symplectic capacity

$$c^{(0)}(M,\omega):=c^{(k)}(M,\omega;pt,\dots,pt).$$

In this paper we shall concentrate on the case $k=2$ since in this case there are interesting examples. More precisely, we shall define a typical G_2-pseudo symplectic capacity of Hofer–Zehnder type and give many applications. In view of our results we expect that pseudo symplectic capacities will become a powerful tool in the study of symplectic topology. Hereafter we assume $G=\mathbb{Q}$ and often write $H_*(M)$ instead of $H_*(M;\mathbb{Q})$.

1.2 Construction of a pseudo symplectic capacity. We begin with recalling the Hofer–Zehnder capacity from [HZ1]. Given a symplectic manifold (M,ω), a smooth function $H\colon M\to\mathbb{R}$ is called **admissible** if there exist a nonempty open subset U and a compact subset $K\subset M\setminus\partial M$ such that

(a) $H|_U=0$ and $H|_{M\setminus K}=\max H$;
(b) $0\le H\le\max H$;
(c) $\dot x=X_H(x)$ has no nonconstant fast periodic solutions.

Here X_H is defined by $\omega(X_H,v)=dH(v)$ for $v\in TM$, and "fast" means "of period less than 1". Let $\mathcal{H}_{ad}(M,\omega)$ be the set of admissible Hamiltonians on (M,ω). The Hofer–Zehnder symplectic capacity $c_{HZ}(M,\omega)$ of (M,ω) is defined by

$$c_{HZ}(M,\omega)=\sup\{\max H|\ H\in\mathcal{H}_{ad}(M,\omega)\}.$$

Note that one can require the compact subset $K=K(H)$ to be a proper subset of M in the definition above. In fact, it suffices to prove that for any $H\in\mathcal{H}_{ad}(M,\omega)$ and $\epsilon>0$ small enough there exists a $H_\epsilon\in\mathcal{H}_{ad}(M,\omega)$ such that $\max H_\epsilon\ge\max H-\epsilon$ and that the corresponding compact subset $K(H_\epsilon)$ is a proper subset in M. Let us take a smooth function $f_\epsilon\colon\mathbb{R}\to\mathbb{R}$ such that $0\le f_\epsilon'(t)\le 1$ and $f_\epsilon(t)=0$ as $t\le 0$, and $f_\epsilon(t)=\max H-\epsilon$ as $t\ge\max H-\epsilon$. Then the composition $f_\epsilon\circ H$ is a desired H_ϵ.

The invariant c_{HZ} has many applications. Three of them are: (i) giving a new proof of a foundational theorem in symplectic topology — Gromov's nonsqueezing theorem; (ii) studying the Hofer geometry on the group of Hamiltonian symplectomorphisms of a symplectic manifold; (iii) establishing the existence of closed characteristics on or near an energy surface. As mentioned above, the difficulties in computing or estimating $c_{HZ}(M,\omega)$ for a given symplectic manifold (M,ω) make it hard to find further applications of this invariant. Therefore, it seems to be important to give a variant of c_{HZ} which can be easily estimated and still has the above applications. An attempt was made in [McSl]. In this paragraph we shall define a pseudo symplectic capacity of Hofer–Zehnder type. The introduction of such a pseudo symplectic capacity was motivated by various papers (e.g., [LiuT, McSl]).

Definition 1.2: For a connected symplectic manifold (M,ω) of dimension at least 4 and two nonzero homology classes $\alpha_0,\alpha_\infty\in H_*(M;\mathbb{Q})$, we call a smooth function $H: M\to\mathbb{R}$ (α_0,α_∞)**-admissible** (resp. $(\alpha_0,\alpha_\infty)^\circ$**-admissible**) if there exist two compact submanifolds P and Q of M with connected smooth boundaries and of codimension zero such that the following condition groups (1)(2)(3)(4)(5)(6) (resp. (1)(2)(3)(4)(5)(6°)) hold:

(1) $P\subset \mathrm{Int}(Q)$ and $Q\subset\mathrm{Int}(M)$.

(2) $H|_P=0$ and $H|_{M\setminus\mathrm{Int}(Q)}=\max H$.

(3) $0\le H\le\max H$.

(4) There exist cycle representatives of α_0 and α_∞, still denoted by α_0,α_∞, such that $\mathrm{supp}(\alpha_0)\subset\mathrm{Int}(P)$ and $\mathrm{supp}(\alpha_\infty)\subset M\setminus Q$.

(5) There are no critical values in $(0,\varepsilon)\cup(\max H-\varepsilon,\max H)$ for a small $\varepsilon=\varepsilon(H)>0$.

(6) The Hamiltonian system $\dot{x}=X_H(x)$ on M has no nonconstant fast periodic solutions;

(6°) The Hamiltonian system $\dot{x}=X_H(x)$ on M has no nonconstant contractible fast periodic solutions.

We respectively denote by

$$\mathcal{H}_{ad}(M,\omega;\alpha_0,\alpha_\infty)\quad\text{and}\quad\mathcal{H}^\circ_{ad}(M,\omega;\alpha_0,\alpha_\infty)\tag{1}$$

the set of all (α_0,α_∞)-admissible and $(\alpha_0,\alpha_\infty)^\circ$-admissible functions. Unlike $\mathcal{H}_{ad}(M,\omega)$ and $\mathcal{H}^\circ_{ad}(M,\omega)$, for some pairs (α_0,α_∞) the sets in (1) might be empty. On the other hand, one easily shows that both sets in (1) are nonempty if α_0 and α_∞ are separated by some hypersurface $S\subset M$ in the following sense.

Definition 1.3: A hypersurface $S \subset M$ is called **separating the homology classes** $\alpha_0, \alpha_\infty \in H_*(M)$ if (i) S separates M in the sense that there exist two submanifolds M_0 and M_∞ of M with common boundary S such that $M_0 \cup M_\infty = M$ and $M_0 \cap M_\infty = S$, (ii) there exist cycle representatives of α_0 and α_∞ with supports contained in $\mathrm{Int}(M_0)$ and $\mathrm{Int}(M_\infty)$ respectively, (iii) M_0 is compact and $\partial M_0 = S$.

Without special statements a hypersurface in this paper always means a smooth compact connected orientable submanifold of codimension one and without boundary. Note that if M is closed and a hypersurface $S \subset M$ separates the homology classes α_0 and α_∞, then S also separates α_∞ and α_0.

We define

$$
(2) \qquad \begin{cases} C^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty) := \sup\{\max H | H \in \mathcal{H}_{ad}(M,\omega;\alpha_0,\alpha_\infty)\}, \\ C^{(2\circ)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty) := \sup\{\max H | \ H \in \mathcal{H}^{\circ}_{ad}(M,\omega;\alpha_0,\alpha_\infty)\}. \end{cases}
$$

Hereafter we make the conventions that $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$. As shown in Theorem 1.5 below, $C^{(2)}_{HZ}$ is a G_2-pseudo symplectic capacity. We call it **pseudo symplectic capacity of Hofer–Zehnder type**. $C^{(2)}_{HZ}$ and $C^{(2\circ)}_{HZ}$ in (2) have similar dynamical implications as the Hofer–Zehnder capacity c_{HZ}. In fact, as in [HZ2, HV2] one shows that

$$
0 < C^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty) < +\infty \quad (0 < C^{(2\circ)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty) < +\infty)
$$

implies that every stable hypersurface $S \subset M$ separating α_0 and α_∞ carries a (contractible in M) closed characteristic, i.e., there is an embedded (contractible in M) circle in S all of whose tangent lines belong to the characteristic line bundle

$$
\mathcal{L}_S = \{(x,\xi) \in TS \mid \omega(\xi,\eta) = 0 \text{ for all } \eta \in T_xS\}.
$$

This leads to the following version of the Weinstein conjecture.

$(\alpha_0, \alpha_\infty)$-Weinstein conjecture: Every hypersurface S of contact type in a symplectic manifold (M,ω) separating α_0 and α_∞ carries a closed characteristic.

In terms of this language the main result Theorem 1.1 in [LiuT] asserts that the $(\alpha_0, \alpha_\infty)$-Weinstein conjecture holds if some GW-invariant

$$
\Psi_{A,g,m+2}(C;\alpha_0,\alpha_\infty,\beta_1,\ldots,\beta_m)
$$

does not vanish; see 1.3 below.

As before, let pt denote the generator of $H_0(M;\mathbb{Q})$ represented by a point. Then we have the true symplectic capacities

$$\begin{cases} C_{HZ}(M,\omega) := C^{(2)}_{HZ}(M,\omega;pt,pt), \\ C^{\circ}_{HZ}(M,\omega) := C^{(2\circ)}_{HZ}(M,\omega;pt,pt). \end{cases} \tag{3}$$

Recall that we have also the π_1-sensitive Hofer–Zehnder capacity denoted $\bar{C}_{HZ}$ in [Lu1] and c°_{HZ} in [Schw]. By definitions, it is obvious that $C_{HZ}(M,\omega) \le c_{HZ}(M,\omega)$ and $C^{\circ}_{HZ}(M,\omega) \le c^{\circ}_{HZ}(M,\omega)$ for any symplectic manifold (M,ω). One naturally asks when C_{HZ} (resp. C°_{HZ}) is equal to c_{HZ} (resp. c°_{HZ}). The following result partially answers this question.

LEMMA 1.4: *Let a symplectic manifold (M,ω) satisfy one of the following conditions:*

(i) *(M,ω) is closed.*

(ii) *For each compact subset $K \subset M\setminus\partial M$ there exists a compact submanifold $W \subset M$ with connected boundary and of codimension zero such that $K \subset W$. Then*

$$C_{HZ}(M,\omega) = c_{HZ}(M,\omega) \quad \text{and} \quad C^{\circ}_{HZ}(M,\omega) = c^{\circ}_{HZ}(M,\omega).$$

For arbitrary homology classes $\alpha_0, \alpha_\infty \in H_*(M)$,

$$\begin{cases} C^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty) \le C^{(2\circ)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty), \\ C^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty) \le C_{HZ}(M,\omega), \\ C^{(2\circ)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty) \le C^{\circ}_{HZ}(M,\omega). \end{cases} \tag{4}$$

Both $C^{(2)}_{HZ}$ and $C^{(2\circ)}_{HZ}$ are important because estimating or calculating them is easier than for C_{HZ} and C°_{HZ}, and because they still share those properties needed for applications. In Remark 1.28 we will give an example which illustrates that sometimes $C^{(2)}_{HZ}$ gives better results than C_{HZ}. Recall that the Gromov width $\mathcal{W}_G$ is the smallest symplectic capacity so that

$$\mathcal{W}_G \le C_{HZ} \le C^{\circ}_{HZ}. \tag{5}$$

Convention: C stands for both $C^{(2)}_{HZ}$ and $C^{(2\circ)}_{HZ}$ if there is no danger of confusion.

The following theorem shows that $C^{(2)}_{HZ}$ is indeed a pseudo symplectic capacity.

THEOREM 1.5:

(i) *If M is closed then for any nonzero homology classes $\alpha_0, \alpha_\infty \in H_*(M;\mathbb{Q})$,*

$$C(M,\omega;\alpha_0,\alpha_\infty) = C(M,\omega;\alpha_\infty,\alpha_0).$$

(ii) $C(M,\omega;\alpha_0,\alpha_\infty)$ *is invariant under those symplectomorphisms* $\psi \in \mathrm{Symp}(M,\omega)$ *which induce the identity on* $H_*(M;\mathbb{Q})$.

(iii) **(Normality)** *For any* $r > 0$ *and nonzero* $\alpha_0, \alpha_\infty \in H_*(B^{2n}(r);\mathbb{Q})$ *or* $H_*(Z^{2n}(r);\mathbb{Q})$,

$$C(B^{2n}(r),\omega_0;\alpha_0,\alpha_\infty) = C(Z^{2n}(r),\omega_0;\alpha_0,\alpha_\infty) = \pi r^2.$$

(iv) **(Conformality)** *For any nonzero real number* λ,

$$C(M,\lambda\omega;\alpha_0,\alpha_\infty) = |\lambda| C(M,\omega;\alpha_0,\alpha_\infty).$$

(v) **(Pseudo monotonicity)** *For any symplectic embedding* $\psi\colon (M_1,\omega_1) \to (M_2,\omega_2)$ *of codimension zero and any nonzero* $\alpha_0,\alpha_\infty \in H_*(M_1;\mathbb{Q})$,

$$C^{(2)}_{HZ}(M_1,\omega_1;\alpha_0,\alpha_\infty) \le C^{(2)}_{HZ}(M_2,\omega_2;\psi_*(\alpha_0),\psi_*(\alpha_\infty)).$$

Furthermore, if ψ *induces an injective homomorphism* $\pi_1(M_1) \to \pi_1(M_2)$ *then*

$$C^{(2\circ)}_{HZ}(M_1,\omega_1;\alpha_0,\alpha_\infty) \le C^{(2\circ)}_{HZ}(M_2,\omega_2;\psi_*(\alpha_0),\psi_*(\alpha_\infty)).$$

(vi) *For any* $m \in \mathbb{N}$

$$\begin{cases} C(M,\omega;\alpha_0,\alpha_\infty) \le C(M,\omega;m\alpha_0,\alpha_\infty), \\ C(M,\omega;\alpha_0,\alpha_\infty) \le C(M,\omega;\alpha_0,m\alpha_\infty), \\ C(M,\omega;-\alpha_0,\alpha_\infty) = C(M,\omega;\alpha_0,\alpha_\infty) = C(M,\omega;\alpha_0,-\alpha_\infty). \end{cases}$$

(vii) *If* $\dim\alpha_0 + \dim\alpha_\infty \le \dim M - 2$ *and* α_0 *or* α_∞ *can be represented by a connected closed submanifold, then*

$$C(M,\omega;\alpha_0,\alpha_\infty) > 0.$$

Remark 1.6: If M is not closed, $C(M,\omega;pt,\alpha)$ and $C(M,\omega;\alpha,pt)$ might be different. For example, let M be the annulus in $\mathbb{R}^2$ of area 2, and α be a generator of $H_1(M)$. Then $\mathcal{W}_G(M,\omega) = C^{(2)}_{HZ}(M,\omega;pt,\alpha) = 2$, while $C^{(2)}_{HZ}(M,\omega;\alpha,pt) = 0$ since $\mathcal{H}_{ad}(M,\omega;\alpha,pt) = \emptyset$. This example also shows that the dimension assumption $\dim\alpha_0 + \dim\alpha_\infty \le \dim M - 2$ cannot be weakened.

Proposition 1.7: *Let* $W \subset \mathrm{Int}(M)$ *be a smooth compact submanifold of codimension zero and with connected boundary such that the homology classes* $\alpha_0,\alpha_\infty \in H_*(M;\mathbb{Q}) \setminus \{0\}$ *have representatives supported in* $\mathrm{Int}(W)$ *and* $\mathrm{Int}(M) \setminus W$, *respectively. Denote by* $\tilde\alpha_0 \in H_*(W;\mathbb{Q})$ *and* $\tilde\alpha_\infty \in H_*(M \setminus W;\mathbb{Q})$ *the nonzero homology classes determined by them. Then*

$$C^{(2)}_{HZ}(W,\omega;\tilde\alpha_0,pt) \le C^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty), \tag{6}$$

and we specially have

$$c_{HZ}(W,\omega) = C_{HZ}(W,\omega) \le C^{(2)}_{HZ}(M,\omega; pt, \alpha) \tag{7}$$

for any $\alpha \in H_*(M;\mathbb{Q})\setminus\{0\}$ *with representative supported in* $\operatorname{Int}(M)\setminus W$. *If the inclusion* $W \hookrightarrow M$ *induces an injective homomorphism* $\pi_1(W) \to \pi_1(M)$ *then*

$$C^{(2\circ)}_{HZ}(W,\omega; \tilde{\alpha}_0, pt) \le C^{(2\circ)}_{HZ}(M,\omega; \alpha_0, \alpha_\infty), \tag{8}$$

and corresponding to (7) we have

$$c^{\circ}_{HZ}(W,\omega) = C^{\circ}_{HZ}(W,\omega) \le C^{(2\circ)}_{HZ}(M,\omega; pt, \alpha). \tag{9}$$

Also

$$C^{(2)}_{HZ}(M\setminus W,\omega; \tilde{\alpha}_\infty, pt) \le C^{(2)}_{HZ}(M,\omega; \alpha_\infty, \alpha_0), \tag{10}$$

and

$$C^{(2\circ)}_{HZ}(M\setminus W,\omega; \tilde{\alpha}_\infty, pt) \le C^{(2\circ)}_{HZ}(M,\omega; \alpha_\infty, \alpha_0) \tag{11}$$

if the inclusion $M\setminus W \hookrightarrow M$ *induces an injective homomorphism* $\pi_1(M\setminus W) \to \pi_1(M)$. *Furthermore, for any* $\alpha \in H_*(M;\mathbb{Q})\setminus\{0\}$ *with* $\dim\alpha \le \dim M - 1$,

$$\mathcal{W}_G(M,\omega) \le C(M,\omega; pt, \alpha). \tag{12}$$

For closed symplectic manifolds, Proposition 1.7 can be strengthened as follows.

THEOREM 1.8: *If in the situation of Proposition 1.7 the symplectic manifold* (M,ω) *is closed and* $M\setminus W$ *is connected, then*

$$C^{(2)}_{HZ}(W,\omega; \tilde{\alpha}_0, pt) + C^{(2)}_{HZ}(M\setminus W,\omega; \tilde{\alpha}_\infty, pt) \le C^{(2)}_{HZ}(M,\omega; \alpha_0, \alpha_\infty). \tag{13}$$

In particular, if $\alpha \in H_*(M;\mathbb{Q})\setminus\{0\}$ *has a representative supported in* $M\setminus W$ *and thus determines a homology class* $\tilde{\alpha} \in H_*(M\setminus W;\mathbb{Q})\setminus\{0\}$, *then*

$$c_{HZ}(W,\omega) + C^{(2)}_{HZ}(M\setminus W,\omega; \tilde{\alpha}, pt) \le C^{(2)}_{HZ}(M,\omega; pt, \alpha).$$

If both inclusions $W \hookrightarrow M$ *and* $M\setminus W \hookrightarrow M$ *induce an injective homomorphisms* $\pi_1(W) \to \pi_1(M)$ *and* $\pi_1(M\setminus W) \to \pi_1(M)$, *then*

$$C^{(2\circ)}_{HZ}(W,\omega; \tilde{\alpha}_0, pt) + C^{(2\circ)}_{HZ}(M\setminus W,\omega; \tilde{\alpha}_\infty, pt) \le C^{(2\circ)}_{HZ}(M,\omega; \alpha_0, \alpha_\infty), \tag{14}$$

and specially

$$c^{\circ}_{HZ}(W,\omega) + C^{(2\circ)}_{HZ}(M \setminus W, \omega; \tilde{\alpha}, pt) \le C^{(2\circ)}_{HZ}(M, \omega; pt, \alpha)$$

for any $\alpha \in H_*(M;\mathbb{Q}) \setminus \{0\}$ *with a representative supported in* $M \setminus W$.

An inequality similar to (13) was first proved for the usual Hofer–Zehnder capacity by Mei-Yue Jiang [Ji]. In the following subsections we always take $G = \mathbb{Q}$.

1.3 Estimating the pseudo capacity in terms of Gromov–Witten invariants. To state our main results we recall that for a given class $A \in H_2(M;\mathbb{Z})$ the Gromov–Witten invariant of genus g and with $m+2$ marked points is a homomorphism

$$\Psi_{A,g,m+2}\colon H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q}) \times H_*(M;\mathbb{Q})^{m+2} \to \mathbb{Q}.$$

We refer to the appendix and [FO, LiT, R, Sie] and [Lu8] for more details on Gromov–Witten invariants.

The Gromov–Witten invariants for general (closed) symplectic manifolds were constructed by different methods; cf. [FO, LiT, R, Sie], and [LiuT] for a Morse theoretic set-up. It is believed that these methods define the same symplectic Gromov–Witten invariants, but no proof has been written down so far. A detailed construction of the GW-invariants by the method in [LiuT], including proofs of the composition law and reduction formula, was given in [Lu8] for a larger class of symplectic manifolds including all closed symplectic manifolds. The method by Liu–Tian was also used in [Mc2]. Without special statements, the Gromov–Witten invariants in this paper are the ones constructed by the method in [LiuT]. The author strongly believes that they agree with those constructed in [R].

Definition 1.9: Let (M,ω) be a closed symplectic manifold and let $\alpha_0, \alpha_\infty \in H_*(M;\mathbb{Q})$. We define

$$\mathrm{GW}_g(M,\omega;\alpha_0,\alpha_\infty) \in (0,+\infty]$$

as the infimum of the ω-areas $\omega(A)$ of the homology classes $A \in H_2(M;\mathbb{Z})$ for which the Gromov–Witten invariant $\Psi_{A,g,m+2}(C;\alpha_0,\alpha_\infty,\beta_1,\dots,\beta_m) \neq 0$ for some homology classes $\beta_1,\dots,\beta_m \in H_*(M;\mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q})$ and an integer $m \ge 1$. We define

$$\mathrm{GW}(M,\omega;\alpha_0,\alpha_\infty) := \inf\{\mathrm{GW}_g(M,\omega;\alpha_0,\alpha_\infty) |\ g \ge 0\} \in [0,+\infty].$$

The positivity $\mathrm{GW}_g(M,\omega;\alpha_0,\alpha_\infty)>0$ follows from the compactness of the space of J-holomorphic stable maps (cf. [FO, LiT, R, Sie]). Here we have used the convention $\inf\emptyset=+\infty$ below (2). One easily checks that both GW_g and GW satisfy the pseudo monotonicity and conformality in Definition 1.1. As Professor Dusa McDuff suggested, one can consider closed symplectic manifolds only and replace the nontriviality condition in Definition 1.1 by

$$c^{(2)}(\mathbb{C}P^n,\sigma_n;pt,pt)=c^{(2)}(\mathbb{C}P^1\times T^{2n-2},\sigma_1\oplus\omega_0;pt,[pt\times T^{2n-2}])=\pi;$$

then both GW_0 and GW are pseudo symplectic capacities in view of (19) and (23) below. The following result is the core of this paper. Its proof is given in §3 based on [LiuT] and the key Lemma 3.3.

THEOREM 1.10: *For any closed symplectic manifold (M,ω) of dimension $\dim M\geq 4$ and homology classes $\alpha_0,\alpha_\infty\in H_*(M;\mathbb{Q})\setminus\{0\}$ we have*

$$C^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty)\leq \mathrm{GW}(M,\omega;\alpha_0,\alpha_\infty) \tag{15}$$

and

$$C^{(2\circ)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty)\leq \mathrm{GW}_0(M,\omega;\alpha_0,\alpha_\infty). \tag{16}$$

Remark 1.11: By the reduction formula (57) for Gromov–Witten invariants recalled in the appendix,

$$\begin{aligned}\Psi_{A,g,m+3}([\pi^{-1}_{m+3}(K)];&\alpha_0,\alpha_\infty,\alpha,\beta_1,\dots,\beta_m)\\ &=PD(\alpha)(A)\cdot\Psi_{A,g,m+2}([K];\alpha_0,\alpha_\infty,\beta_1,\dots,\beta_m)\end{aligned}$$

for any $\alpha\in H_{2n-2}(M,\mathbb{Z})$ and $[K]\in H_*(\overline{\mathcal{M}}_{g,m+2},\mathbb{Q})$. Here $2n=\dim M$. It easily follows that $\mathrm{GW}_g(M,\omega;\alpha_0,\alpha_\infty)<+\infty$ implies that $\mathrm{GW}_g(M,\omega;\alpha_0,\alpha)$, $\mathrm{GW}_g(M,\omega;\alpha,\alpha_\infty)$ and $\mathrm{GW}_g(M,\omega;\alpha,\beta)$ are finite for any $\alpha,\beta\in H_{2n-2}(M,\mathbb{Z})$ with $PD(\alpha)(A)\neq 0$ and $PD(\beta)(A)\neq 0$. In particular, it is easily proved that for any integer $g\geq 0$

$$\mathrm{GW}_g(M,\omega;pt,PD([\omega]))=\inf\{\mathrm{GW}_g(M,\omega;pt,\alpha)|\ \alpha\in H_*(M,\mathbb{Q})\}. \tag{17}$$

COROLLARY 1.12: *If $\mathrm{GW}_g(M,\omega;\alpha_0,\alpha_\infty)<+\infty$ for some integer $g\geq 0$ then the (α_0,α_∞)-Weinstein conjecture holds in (M,ω).*

Many results in this paper are based on the following special case of Theorem 1.10.

THEOREM 1.13: *For any closed symplectic manifold (M, ω) of dimension at least four and a nonzero homology class $\alpha \in H_*(M; \mathbb{Q})$,*

$$C_{HZ}^{(2)}(M, \omega; pt, \alpha) \leq \mathrm{GW}(M, \omega; pt, \alpha)$$

and

$$C_{HZ}^{(2\circ)}(M, \omega; pt, \alpha) \leq \mathrm{GW}_0(M, \omega; pt, \alpha).$$

Definition 1.14: Given a nonnegative integer g, a closed symplectic manifold (M, ω) is called **g-symplectic uniruled** if $\Psi_{A,g,m+2}(C; pt, \alpha, \beta_1, \ldots, \beta_m) \neq 0$ for some homology classes $A \in H_2(M; \mathbb{Z})$, $\alpha, \beta_1, \ldots, \beta_m \in H_*(M; \mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+2}; \mathbb{Q})$ and an integer $m \geq 1$. If C can be chosen as a point pt we say (M, ω) is **strong g-symplectic uniruled**. Moreover, (M, ω) is called **symplectic uniruled** (resp. **strong symplectic uniruled**) if it is g-symplectic uniruled (resp. strong g-symplectic uniruled) for some integer $g \geq 0$.

It was proved in ([Ko]) and ([R]) that (projective algebraic) uniruled manifolds are strong 0-symplectic uniruled.* In Proposition 7.3 we shall prove that for a closed symplectic manifold (M, ω), if there exist homology classes $A \in H_2(M; \mathbb{Z})$ and $\alpha_i \in H_*(M; \mathbb{Q})$, $i = 1, \ldots, k$, such that the Gromov–Witten invariant $\Psi_{A,g,k+1}(pt; pt, \alpha_1, \ldots, \alpha_k) \neq 0$ for some integer $g \geq 0$, then there exists a homology class $B \in H_2(M; \mathbb{Z})$ with $\omega(B) \leq \omega(A)$ and $\beta_i \in H_*(M; \mathbb{Q})$, $i = 1, 2$, such that the Gromov–Witten invariant $\Psi_{B,0,3}(pt; pt, \beta_1, \beta_2) \neq 0$. Therefore, every strong symplectic uniruled manifold is strong 0-symplectic uniruled. Actually, we shall prove in Proposition 7.5 that the product of any closed symplectic manifold and a strong symplectic uniruled manifold is strong symplectic uniruled. Moreover, the class of g-symplectic uniruled manifolds is closed under deformations of symplectic forms because Gromov–Witten invariants are symplectic deformation invariants. For a g-symplectic uniruled manifold (M, ω), i.e., $\mathrm{GW}_g(M, \omega; pt, PD([\omega])) < +\infty$, the author observed in [Lu3] that if a hypersurface of contact type S in (M, ω) separates M into two parts M_+ and M_-, then there exist two classes $PD([\omega])_+$ and $PD([\omega])_-$ in $H_{2n-2}(M, \mathbb{R})$ with cycle representatives supported in M_+ and M_- respectively such that $PD([\omega])_+ + PD([\omega])_- = PD([\omega])$ and that at least one of the numbers $\mathrm{GW}_g(M, \omega; pt, PD([\omega])_+)$ or $\mathrm{GW}_g(M, \omega; pt, PD([\omega])_-)$ is finite. Theorem 1.13

* This is the only place in which we assume that our GW-invariants agree with the ones in [R]. In a future paper we shall use the method in [LiuT] and the techniques in [Lu8] to prove this fact.

(or (15)) implies that at least one of the following two statements holds:

$$(18)\qquad \begin{aligned} &C^{(2)}_{HZ}(M,\omega;pt,PD([\omega])_+)\le \mathrm{GW}_g(M,\omega;pt,PD([\omega])_+)<+\infty \quad\text{or}\\ &C^{(2)}_{HZ}(M,\omega;pt,PD([\omega])_-)\le \mathrm{GW}_g(M,\omega;pt,PD([\omega])_-)<+\infty.\end{aligned}$$

On the other hand, (12) shows that $C^{(2)}_{HZ}(M,\omega;pt,PD([\omega])_+)$ and $C^{(2)}_{HZ}(M,\omega;pt,PD([\omega])_-)$ are always positive. Consequently, S carries a nontrivial closed characteristic, i.e., the (pt,pt)-Weinstein conjecture holds in symplectic uniruled manifolds ([Lu3]).

The Grassmannians and their products with any closed symplectic manifold are symplectic uniruled. For them we have

THEOREM 1.15: *For the Grassmannian $G(k,n)$ of k-planes in $\mathbb{C}^n$ we denote by $\sigma^{(k,n)}$ the canonical symplectic form for which $\sigma^{(k,n)}(L^{(k,n)})=\pi$ for the generator $L^{(k,n)}$ of $H_2(G(k,n);\mathbb{Z})$. Let the submanifolds $X^{(k,n)}\approx G(k,n-1)$ and $Y^{(k,n)}$ of $G(k,n)$ be given by $\{V\in G(k,n)\mid w_0^*v=0 \text{ for all } v\in V\}$ and $\{V\in G(k,n)\mid v_0\in V\}$ for some fixed $v_0,w_0\in\mathbb{C}^n\setminus\{0\}$ respectively. Their homology classes $[X^{(k,n)}]$ and $[Y^{(k,n)}]$ are independent of the choices of $v_0,w_0\in\mathbb{C}^n\setminus\{0\}$ and $\deg[X^{(k,n)}]=2k(n-k-1)$ and $\deg[Y^{(k,n)}]=2(k-1)(n-k)$. Then*

$$\mathcal{W}_G(G(k,n),\sigma^{(k,n)})=C^{(2)}_{HZ}(G(k,n),\sigma^{(k,n)};pt,\alpha)=\pi$$

for $\alpha=[X^{(k,n)}]$ or $\alpha=[Y^{(k,n)}]$ with $k\le n-2$.

In particular, if $k=1$ and $n\ge 3$ then $[Y^{(1,n)}]=pt$ and $(G(1,n),\sigma^{(1,n)})=(\mathbb{C}P^{n-1},\sigma_{n-1})$, where σ_{n-1} the unique U(n)-invariant Kähler form on $\mathbb{C}P^{n-1}$ whose integral over the line $\mathbb{C}P^1\subset\mathbb{C}P^{n-1}$ is equal to π. In this case Theorem 1.15 and Lemma 1.4 yield

$$(19)\qquad \begin{aligned} c_{HZ}(\mathbb{C}P^{n-1},\sigma_{n-1})&=C_{HZ}(\mathbb{C}P^{n-1},\sigma_{n-1})\\ &:=C^{(2)}_{HZ}(\mathbb{C}P^{n-1},\sigma_{n-1};pt,pt)=\pi.\end{aligned}$$

Hofer and Viterbo [HV2] first proved that $c_{HZ}(\mathbb{C}P^n,\sigma_n)=\pi$. Therefore, Theorem 1.15 can be viewed as a generalization of their result. If $k=1$, on one hand the volume estimate gives $\mathcal{W}_G(\mathbb{C}P^{n-1},\sigma_{n-1})\le\pi$, and on the other hand there exists an explicit symplectic embedding $B^{2n-2}(1)\hookrightarrow(\mathbb{C}P^{n-1},\sigma_{n-1})$; see [Ka, HV2]. So we have $\mathcal{W}_G(\mathbb{C}P^{n-1},\sigma_{n-1})=\pi$. For $k\ge 2$, however, the remarks below Theorem 1.35 show that the identity $\mathcal{W}_G(G(k,n),\sigma^{(k,n)})=\pi$ does not follow so easily. Karshon and Tolman [KaTol] independently computed $\mathcal{W}_G(G(k,n),\sigma^{(k,n)})$ in a different method.

THEOREM 1.16: *For any closed symplectic manifold* (M,ω),

$$C(M\times G(k,n),\omega\oplus(a\sigma^{(k,n)});pt,[M]\times\alpha)\le|a|\pi \tag{20}$$

for any $a\in\mathbb{R}\setminus\{0\}$ *and* $\alpha=[X^{(k,n)}]$ or $\alpha=[Y^{(k,n)}]$ *with* $k\le n-2$. *Moreover, for the product*

$$(W,\Omega)=(G(k_1,n_1)\times\cdots\times G(k_r,n_r),(a_1\sigma^{(k_1,n_1)})\oplus\cdots\oplus(a_r\sigma^{(k_r,n_r)}))$$

we have

$$C(W,\Omega;pt,\alpha_1\times\cdots\times\alpha_r)\le(|a_1|+\cdots+|a_r|)\pi \tag{21}$$

for any $a_i\in\mathbb{R}\setminus\{0\}$ *and* $\alpha_i=[X^{(k_i,n_i)}]$ or $[Y^{(k_i,n_i)}]$. *Furthermore,*

$$\mathcal{W}_G(G(k_1,n_1)\times\cdots\times G(k_r,n_r),\sigma^{(k_1,n_1)}\oplus\cdots\oplus\sigma^{(k_r,n_r)})=\pi \tag{22}$$

For the projective space $\mathbb{C}P^n=G(1,n+1)$ we have

THEOREM 1.17: *Let* (M,ω) *be a closed symplectic manifold and* σ_n *the unique* $\mathrm{U}(n+1)$*-invariant Kähler form on* $\mathbb{C}P^n$ *whose integral over the line* $\mathbb{C}P^1\subset\mathbb{C}P^n$ *is equal to* π. *Then*

$$C(M\times\mathbb{C}P^n,\omega\oplus(a\sigma_n);pt,[M\times pt])=|a|\pi \tag{23}$$

for any $a\in\mathbb{R}\setminus\{0\}$. *Moreover, for any* $r>0$ *and the standard ball* $B^{2n}(r)$ *of radius* r *and the cylinder* $Z^{2n}(r)=B^2(r)\times\mathbb{R}^{2n-2}$ *in* $(\mathbb{R}^{2n},\omega_0)$, *we have*

$$C(M\times B^{2n}(r),\omega\oplus\omega_0)=C(M\times Z^{2n}(r),\omega\oplus\omega_0)=\pi r^2 \tag{24}$$

for $C=C_{HZ}$, C°_{HZ}, c_{HZ} and c°_{HZ}.

Remark 1.18: Combining the arguments in [McSl, Lu1] one can prove a weaker version of (24) for any weakly monotone noncompact geometrically bounded symplectic manifold (M,ω) and any $r>0$, namely

$$C^\circ_{HZ}(M\times B^{2n}(r),\omega\oplus\omega_0)\le C^\circ_{HZ}(M\times Z^{2n}(r),\omega\oplus\omega_0)\le\pi r^2.$$

This generalization can be used to find periodic orbits of a charge subject to a magnetic field (cf. [Lu2]).

From Theorem 1.13 and Lemma 1.4 we obtain

COROLLARY 1.19: *For any closed symplectic manifold (M,ω) of dimension at least 4 we have*

$$c_{HZ}(M,\omega) \le \mathrm{GW}(M,\omega;pt,pt), \quad c^\circ_{HZ}(M,\omega) \le \mathrm{GW}_0(M,\omega;pt,pt).$$

Thus $c_{HZ}(M,\omega)$ is finite if the Gromov–Witten invariant

$$\Psi_{A,g,m+2}(C;pt,pt,\beta_1,\dots,\beta_m)$$

does not vanish for some homology classes $A \in H_2(M;\mathbb{Z})$, $\beta_1,\dots,\beta_m \in H_*(M;\mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q})$ and integers $g \ge 0$ and $m > 0$. Notice that $\mathrm{GW}_0(M,\omega;pt,pt)$ is needed here. For example, consider

$$(M,\omega) = (\mathbb{C}P^1 \times \mathbb{C}P^1, \sigma_1 \oplus \sigma_1).$$

The following Theorem 1.21 and its proof show that $c_{HZ}(M,\omega) = c^\circ_{HZ}(M,\omega) = 2\pi$ and $\mathrm{GW}_0(M,\omega;pt,pt) = 2\pi$. However, one easily proves that

$$\begin{aligned}\mathrm{GW}_0(M,\omega;pt,PD([\omega])) &= \mathrm{GW}_0(M,\omega;pt,[pt \times \mathbb{C}P^1]) \\ &= \mathrm{GW}_0(M,\omega;pt,[\mathbb{C}P^1 \times pt]) = \pi.\end{aligned}$$

So $\mathrm{GW}_0(M,\omega;pt,pt)$ is necessary.

Example 1.20: (i) For a smooth complete intersection (X,ω) of degree $(d_1,\dots,d_k)$ in $\mathbb{C}P^{n+k}$ with $n = 2\sum(d_i-1)-1$ or $3\sum(d_i-1)-3$, we have $c^\circ_{HZ}(X,\omega) = C^\circ_{HZ}(X,\omega) < +\infty$.

(ii) For a rational algebraic manifold (X,ω), if there exists a surjective morphism $\pi\colon X \to \mathbb{C}P^n$ such that $\pi|_{X\setminus S}$ is one-to-one for some subvariety S of X with $\mathrm{codim}_{\mathbb{C}}\,\pi(S) \ge 2$, then $c^\circ_{HZ}(X,\omega) = C^\circ_{HZ}(X,\omega)$ is finite.

(i) follows from the corollaries of Propositions 3 and 4 in [Be] and (ii) comes from Theorem 1.5 in [LiuT]. We conjecture that the conclusion also holds for the **rationally connected manifolds** introduced in [KoMiMo].

In some cases we can get better results.

THEOREM 1.21: *For the standard symplectic form σ_{n_i} on $\mathbb{C}P^{n_i}$ as in Theorem 1.17 and any $a_i \in \mathbb{R}\setminus\{0\}$, $i = 1,\dots,k$, we have*

$$C(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}, a_1\sigma_{n_1} \oplus \cdots \oplus a_k\sigma_{n_k}) = (|a_1| + \cdots + |a_k|)\pi$$

for $C = c_{HZ}$ and c°_{HZ}.

According to Example 12.5 of [McSal]

$$\mathcal{W}_G(\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1, a_1\sigma_1 \oplus \cdots \oplus a_k\sigma_1) = \min\{|a_1|,\dots,|a_k|\}\pi$$

for any $a_i \in \mathbb{R} \setminus \{0\}$, $i = 1, \dots, k$. This, Theorem 1.21 and (5) show that C_{HZ}, C°_{HZ}, c_{HZ} and c°_{HZ} are different from the Gromov width $\mathcal{W}_G$.

1.4 The Weinstein conjecture and periodic orbits near symplectic submanifolds.

1.4.1. The Weinstein conjecture in cotangent bundles of uniruled manifolds. By "The Weinstein conjecture" we in the sequel mean the (pt, pt)-Weinstein conjecture, i.e.: Every separating hypersurface S of contact type in a symplectic manifold carries a closed characteristic. While in some of the previous works on the Weinstein conjecture, e.g. [HV1], the assumption that S is separating was also imposed, Weinstein's original conjecture, [We2], does not assume that S is separating. So far this conjecture has been proved for many symplectic manifolds; cf. [C, FHV, FrSchl, H2, HV1, HV2, LiuT, Lu1, Lu2, Lu3, V1, V4, V5] and the recent nice survey [Gi] for more references. In particular, for the Weinstein conjecture in cotangent bundles Hofer and Viterbo [HV1] proved that if a connected hypersurface S of contact type in the cotangent bundle of a closed manifold N of dimension at least 2 is such that the bounded component of $T^*N \setminus S$ contains the zero section of T^*N, then it carries a closed characteristic. In [V5] it was proved that the Weinstein conjecture holds in cotangent bundles of simply connected closed manifolds. We shall prove

Theorem 1.22: *Let (M, ω) be a closed connected symplectic manifold of dimension at least 4 and let $L \subset M$ be a Lagrangian submanifold. Given a homology class $\tilde{\alpha}_0 \in H_*(L; \mathbb{Q}) \setminus \{0\}$ we denote by $\alpha_0 \in H_*(M; \mathbb{Q})$ the class induced by the inclusion $L \hookrightarrow M$. Assume that the Gromov–Witten invariant $\Psi_{A,g,m+1}(C; \alpha_0, \alpha_1, \dots, \alpha_m)$ does not vanish for some homology classes $A \in H_2(M; \mathbb{Z})$, $\alpha_1, \dots, \alpha_m \in H_*(M; \mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+1}; \mathbb{Q})$ and integers $m > 1$ and $g > 0$. Then for every $c > 0$, $C^{(2)}_{HZ}(U_c, \omega_{\text{can}}; \tilde{\alpha}_0, pt) < +\infty$, and*

$$C^{(2\circ)}_{HZ}(U_c, \omega_{\text{can}}; \tilde{\alpha}_0, pt) < +\infty$$

if $g = 0$ and the inclusion $L \hookrightarrow M$ induces an injective homomorphism $\pi_1(L) \to \pi_1(M)$. Here $U_c = \{(q, v^*) \in T^*L |\ \langle v^*, v^* \rangle \le c^2\}$ is with respect to a Riemannian metric $\langle \cdot, \cdot \rangle$ on T^*L. Consequently, every hypersurface of contact type in $(T^*L, \omega_{\text{can}})$ separating $\tilde{\alpha}_0$ and pt carries a closed characteristic and a contractible one in the latter case. In particular, if (M, ω) is a g-symplectic uniruled manifold then for each $c > 0$

$$c_{HZ}(U_c, \omega_{\text{can}}) = C_{HZ}(U_c, \omega_{\text{can}}) < +\infty$$

and

$$c^{\circ}_{HZ}(U_c, \omega_{\rm can}) = C^{\circ}_{HZ}(U_c, \omega_{\rm can}) < +\infty \tag{25}$$

if $g = 0$ *and the inclusion* $L \hookrightarrow M$ *induces an injective homomorphism* $\pi_1(L) \to \pi_1(M)$. *If* (M, ω) *itself is strong symplectic uniruled then* (25) *also holds for* $L = M \subset (T^*M, \omega_{\rm can})$.

Using a recent refinement by Macarini and Schlenk [MaSchl] of the arguments in [HZ2, Sections 4.1 and 4.2] we immediately derive: if L is a Lagrangian submanifold in a g-symplectic uniruled manifold and $S \subset (T^*L, \omega_{\rm can})$ a smooth compact connected orientable hypersurface without boundary, then for any thickening of S,

$$\psi: I \times S \to U \subset (T^*L, \omega_{\rm can}),$$
$$\mu\{t \in I |\, \mathcal{P}(S_t) \neq \emptyset\} = \mu(I) \quad \text{and} \quad \mu\{t \in I |\, \mathcal{P}^{\circ}(S_t) \neq \emptyset\} = \mu(I)$$

if $g = 0$ and the inclusion $L \hookrightarrow M$ induces an injective homomorphism $\pi_1(L) \to \pi_1(M)$. Here μ denotes Lebesgue measure, I is an open neighborhood of 0 in $\mathbb{R}$, and $\mathcal{P}(S_t)$ (resp. $\mathcal{P}^{\circ}(S_t)$) denotes the set of all (resp. contractible in U) closed characteristics on $S_t = \psi(S \times \{t\})$.

COROLLARY 1.23: *The Weinstein conjecture holds in the following manifolds:*

(i) *symplectic uniruled manifolds of dimension at least* 4;
(ii) *the cotangent bundle* $(T^*L, \omega_{\rm can})$ *of a closed Lagrangian submanifold* L *in a* g*-symplectic uniruled manifold of dimension at least* 4;
(iii) *the product of a closed symplectic manifold and a strong symplectic uniruled manifold;*
(iv) *the cotangent bundles of strong symplectic uniruled manifolds.*

The result in (i) is actually not new. As observed in [Lu3] the Weinstein conjecture in symplectic uniruled manifolds can be derived from Theorem 1.1 in [LiuT]. With the present arguments it may be derived from (18) and Corollary 1.12. (ii) is a direct consequence of Theorem 1.22. (iii) can be derived from (i) and Proposition 7.5. By (ii) and Proposition 7.5 the standard arguments give rise to (iv).

1.4.2. Periodic orbits near symplectic submanifolds. The existence of periodic orbits of autonomous Hamiltonian systems near a closed symplectic submanifold has been studied by several authors; see [CiGiKe, GiGu, Ke] and the references there for details. Using Proposition 1.7 and suitably modifying with the arguments in [Lu6] and [Bi1] we get

THEOREM 1.24: *Let (M, w) be any symplectic manifold and let $N \subset M$ be a connected closed symplectic submanifold of codimension 2. Then for any $\varepsilon > 0$ there exists a smooth compact submanifold $W \subset M$ with connected boundary and of codimension zero which is a neighborhood of N in M such that*

$$c^{\circ}_{HZ}(W, w) = C^{\circ}_{HZ}(W, w) < \varepsilon.$$

Consequently, for any smooth compact connected orientable hypersurface $S \subset W \setminus \partial W$ without boundary and any thickening ψ: $S \times I \to U \subset W$ it holds that

$$\mu(\{t \in I | \mathcal{P}^{\circ}(S_t) \neq \emptyset\}) = \mu(I).$$

Here μ, I, S_t and $\mathcal{P}^{\circ}(S_t)$ are as above Corollary 1.23.

The first conclusion will be proved in §5, and the second follows from the first one and the refinement of the Hofer–Zehnder theorem by Macarini and Schlenk [MaSchl] mentioned above. The second conclusion in Theorem 1.24 implies: For any smooth proper function H: $W \to \mathbb{R}$ the levels $H = \epsilon$ carry contractible in U periodic orbits for almost all $\epsilon > 0$ for which $\{H = \epsilon\} \subset \mathrm{Int}(W)$. Using Floer homology and symplectic homology, results similar to Theorem 1.24 were obtained in [CiGiKe, GiGu] for any closed symplectic submanifolds of positive codimension in geometrically bounded, symplectically aspherical manifolds. Recall that a symplectic manifold (M, ω) is said to be symplectically aspherical if $\omega|_{\pi_2(M)} = 0$ and $c_1(TM)|_{\pi_2(M)} = 0$. It seems possible that our method can be generalized to any closed symplectic submanifold of codimension more than 2.

1.5 NONSQUEEZING THEOREMS. We first give a general nonsqueezing theorem and then discuss some corollaries and relations to the various previously found nonsqueezing theorems.

Definition 1.25: For a symplectic manifold (M, ω) we define $\Gamma(M, \omega) \in [0, +\infty]$ by

$$\Gamma(M, \omega) = \inf_{\alpha} C^{(2)}_{HZ}(M, \omega; pt, \alpha),$$

where $\alpha \in H_*(M; \mathbb{Q})$ runs over all nonzero homology classes of degree $\deg \alpha \leq \dim M - 1$.

By (12), for any connected symplectic manifold (M, ω) we have

$$\mathcal{W}_G(M, \omega) \leq \Gamma(M, \omega). \tag{26}$$

However, it is difficult to determine or estimate $\Gamma(M, \omega)$. In some cases one can replace it by another number.

Definition 1.26: For a closed connected symplectic manifold (M, ω) of dimension at least 4 we define $\mathrm{GW}(M, \omega) \in (0, +\infty]$ by

$$\mathrm{GW}(M, \omega) = \inf \mathrm{GW}_g(M, \omega; pt, \alpha),$$

where the infimum is taken over all nonnegative integers g and all homology classes $\alpha \in H_*(M; \mathbb{Q}) \setminus \{0\}$ of degree $\deg \alpha \leq \dim M - 1$.

By (17) we have $\mathrm{GW}(M, \omega) = \inf_g \mathrm{GW}_g(M, \omega; pt, PD([\omega]))$. Note that $\mathrm{GW}(M, \omega)$ is finite if and only if (M, ω) is a symplectic uniruled manifold. From Theorem 1.13 and (26) we get

THEOREM 1.27: *For any symplectic uniruled manifold (M, ω) of dimension at least 4 we have*

$$\mathcal{W}_G(M, \omega) \leq \mathrm{GW}(M, \omega).$$

Actually, for a uniruled manifold (M, ω), i.e., a Kähler manifold covered by rational curves, the arguments in [Ko, R] show that $\mathrm{GW}(M, \omega) \leq \omega(A)$, where $A = [C]$ is the class of a rational curve C through a generic $x_0 \in M$ and such that $\int_C \omega$ is minimal.

Remark 1.28: Denote by (W, Ω) the product

$$(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}, a_1\sigma_{n_1} \oplus \cdots \oplus a_k\sigma_{n_k})$$

in Theorem 1.21. It follows from Theorem 1.13 and the proof of Theorem 1.17 that

$$\mathrm{GW}(W, \Omega) \leq \min\{|a_1|, \ldots, |a_k|\}\pi.$$

By (26) and definition of $\Gamma(W, \Omega)$, for any small $\epsilon > 0$ there exists a class $\alpha_\epsilon \in H_*(W, \mathbb{Q})$ of degree $\deg(\alpha_\epsilon) \leq \dim W - 1$ such that

$$\mathcal{W}_G(W, \Omega) \leq C_{HZ}^{(2)}(W, \Omega; pt, \alpha_\epsilon) < \min\{|a_1|, \ldots, |a_k|\}\pi + \epsilon.$$

But Theorem 1.21 shows that

$$c_{HZ}(W, \Omega) = C_{HZ}(W, \Omega) = (|a_1| + \cdots + |a_k|)\pi.$$

Therefore, if $k > 1$ and $\epsilon > 0$ is small enough then

$$\mathcal{W}_G(W, \Omega) \leq C_{HZ}^{(2)}(W, \Omega; pt, \alpha_\epsilon) < C_{HZ}(W, \Omega).$$

This shows that our pseudo symplectic capacity $C_{HZ}^{(2)}(W, \Omega; pt, \alpha_\epsilon)$ can give a better upper bound for $\mathcal{W}_G(W, \Omega)$ than the symplectic capacities $c_{HZ}(W, \Omega)$ and $C_{HZ}(W, \Omega)$.

Recall that Gromov's famous nonsqueezing theorem states that if there exists a symplectic embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$, then $r \le R$. Gromov proved it by using J-holomorphic curves, [Gr]. Later on, proofs were given by Hofer and Zehnder based on the calculus of variation and by Viterbo using generating functions, [V3]. As a direct consequence of Theorem 1.5 and (24) we get

COROLLARY 1.29: *For any closed symplectic manifold (M,ω) of dimension $2m$, if there exists a symplectic embedding*

$$B^{2m+2n}(r) \hookrightarrow (M \times Z^{2n}(R), \omega \oplus \omega_0),$$

then $r \le R$.

Actually, Lalonde and McDuff proved Corollary 1.29 for any symplectic manifold (M,ω) in [LaMc1]. Moreover, one can derive from it the foundational energy-capacity inequality in Hofer geometry (cf. [LaMc1, La2] and [McSa1, Ex. 12.21]). From (24) one can also derive the following version of the nonsqueezing theorem which was listed below Corollary 5.8 of [LaMc2,II] and which can be used to prove that the group of Hamiltonian diffeomorphisms of some compact symplectic manifolds have infinite diameter with respect to Hofer's metric.

COROLLARY 1.30: *Let (M,ω) and (N,σ) be closed symplectic manifolds of dimensions $2m$ and $2n$ respectively. If there exists a symplectic embedding*

$$M \times B^{2n+2p}(r) \hookrightarrow (M \times N \times B^{2p}(R), \omega \oplus \sigma \oplus \omega_0^{(p)})$$

or a symplectic embedding

$$M \times B^{2n+2p}(r) \hookrightarrow (M \times \mathbb{R}^{2n} \times B^{2p}(R), \omega \oplus \omega_0^{(n)} \oplus \omega_0^{(p)}),$$

then $r \le R$. Here $\omega_0^{(m)}$ denotes the standard symplectic structure on $\mathbb{R}^{2m}$.

The second statement can be reduced to the first one. From Theorem 1.16 we get

COROLLARY 1.31: *For any closed symplectic manifold (M,ω) of dimension $2m$,*

$$\mathcal{W}_G(M \times G(k,n), \omega \oplus (a\sigma^{(k,n)})) \le |a|\pi.$$

The study of Hofer geometry requires various nonsqueezing theorems. Let us recall the notion of quasicylinder introduced by Lalonde and McDuff in [LaMc2].

Definition 1.32: For a closed symplectic manifold (M, ω) and a set D diffeomorphic to a closed disk in $(\mathbb{R}^2, \omega_0 = ds \wedge dt)$, the manifold $Q = (M \times D, \Omega)$ endowed with the symplectic form Ω is called a **quasicylinder** if

(i) Ω restricts to ω on each fibre $M \times \{pt\}$;

(ii) Ω is the product $\omega \times \omega_0$ near the boundary $\partial Q = M \times \partial D$.

If $\Omega = \omega \times \omega_0$ on Q, the quasicylinder is called **split**. The **area** of a quasicylinder $(M \times D, \Omega)$ is defined as the number $\Lambda = \Lambda(M \times D, \Omega)$ such that

$$\mathrm{Vol}(M \times D, \Omega) = \Lambda \cdot \mathrm{Vol}(M, \omega).$$

As proved in Lemma 2.4 of [LaMc2], the area $\Lambda(M \times D, \Omega)$ is equal to $\int_{\{x\}\times D} \Omega$ for any $x \in M$.

Following [McSl] we replace Q in Definition 1.32 by the obvious S^2-compactification $(M \times S^2, \Omega)$. Here Ω restricts to ω on each fibre. It is clear that $\Omega(A) = \Lambda(Q, \Omega)$ for $A = [pt \times S^2] \in H_2(M \times S^2)$. But it is proved in Lemma 2.7 of [LaMc2] that Ω can be symplectically deformed to a product symplectic form $\omega \oplus \sigma$. Therefore, it follows from the deformation invariance of Gromov–Witten invariants that

$$\Psi_{A,0,3}(pt; pt, [M \times pt], [M \times pt]) \neq 0.$$

By Theorem 1.13 we get

$$C(M \times S^2, \Omega; pt, [M \times pt]) \leq \Omega(A) = \Lambda(Q, \Omega).$$

As in the proof of Theorem 1.17 we can derive from this

THEOREM 1.33 (Area-capacity inequality): *For any quasicylinder* (Q, Ω)

$$c^{\circ}_{HZ}(Q, \Omega) = C^{\circ}_{HZ}(Q, \Omega) \leq \Lambda(Q, \Omega).$$

Area-capacity inequalities for $\mathcal{W}_G$, c_{HZ} and c°_{HZ} have been studied in [FHV, HV2, LaMc1, Lu1, McSl]. As in [LaMc2, McSl] we can use Theorem 1.33 and Lemma 1.4 to deduce the main result in [McSl]: For an autonomous Hamiltonian $H\colon M \to \mathbb{R}$ on a closed symplectic manifold (M, ω) of dimension at least 4, if its flow has no nonconstant contractible fast periodic solution then the path $\phi^H_{t\in[0,1]}$ in $\mathrm{Ham}(M, \omega)$ is length-minimizing among all paths homotopic with fixed endpoints.

From Theorem 1.33 and (5) we obtain the following nonsqueezing theorem for quasi-cylinders.

COROLLARY 1.34: *For any quasicylinder $(M \times D, \Omega)$ of dimension $2m+2$,*

$$\mathcal{W}_G(M \times D, \Omega) \leq \Lambda(M \times D, \Omega).$$

Our results also lead to the nonsqueezing theorem Proposition 3.27 in [Mc2] for Hamiltonian fibrations $P \to S^2$.

1.6 SYMPLECTIC PACKINGS AND SESHADRI CONSTANTS.

1.6.1. Symplectic packings. Suppose that $B^{2n}(r) = \{z \in \mathbb{R}^{2n} | \, |z| < r\}$ is endowed with the standard symplectic structure ω_0 of $\mathbb{R}^{2n}$. For an integer $k > 0$, a **symplectic k-packing** of a $2n$-dimensional symplectic manifold (M, ω) via $B^{2n}(r)$ is a set of symplectic embeddings $\{\varphi_i\}_{i=1}^k$ of $(B^{2n}(r), \omega_0)$ into (M, ω) such that $\operatorname{Im}\varphi_i \cap \operatorname{Im}\varphi_j = \emptyset$ for $i \neq j$. If $\operatorname{Vol}(M, \omega)$ is finite and $\operatorname{Int}(M) \subset \overline{\cup \operatorname{Im}\varphi_i}$, then (M, ω) is said to have a **full symplectic k-packing**. Symplectic packing problems were studied for the first time by Gromov in [Gr] and later by McDuff and Polterovich [McPo], Karshon [Ka], Traynor [Tr], Xu [Xu], Biran [Bi1, Bi2] and Kruglikov [Kru]. As before, let σ_n denote the unique $\mathrm{U}(n+1)$-invariant Kähler form on $\mathbb{C}P^n$ whose integral over $\mathbb{C}P^1$ is equal to π. For every positive integer p, a full symplectic p^n-packing of $(\mathbb{C}P^n, \sigma_n)$ was explicitly constructed by McDuff and Polterovich [McPo] and Traynor [Tr]. A direct geometric construction of a full symplectic $(n+1)$-packing of $(\mathbb{C}P^n, \sigma_n)$ was given by Yael Karshon, [Ka]. By generalizing the arguments in [Ka] we shall obtain

THEOREM 1.35: *Let the Grassmannian $(G(k,n), \sigma^{(k,n)})$ be as in Theorem 1.15. Then for every integer $1 < k < n$ there exists a symplectic $[n/k]$-packing of $(G(k,n), \sigma^{(k,n)})$ by $B^{2k(n-k)}(1)$. Here $[n/k]$ denotes the largest integer less than or equal to n/k.*

This result shows that the Fefferman invariant of $(G(k,n), \sigma^{(k,n)})$ is at least $[n/k]$. Recall that the Fefferman invariant $F(M, \omega)$ of a $2n$-dimensional symplectic manifold (M, ω) is defined as the largest integer k for which there exists a symplectic packing by k open unit balls. Moreover, at the end of §6 we shall prove

$$\text{(27)} \quad \operatorname{Vol}(G(k,n), \sigma^{(k,n)}) = \frac{(k-1)! \cdots 2! \cdot 1! \cdot (n-k-1)! \cdots 2! \cdot 1!}{(n-1)! \cdots 2! \cdot 1!} \cdot \pi^{k(n-k)}.$$

Note that $\operatorname{Vol}(B^{2k(n-k)}(1), \omega_0) = \pi^{k(n-k)}/(k(n-k))!$. One easily sees that the symplectic packing in Theorem 1.35 is not full in general. On the other hand, a full packing of each of the Grassmannians $Gr^+(2,\mathbb{R}^5)$ and $Gr^+(2,\mathbb{R}^6)$ by two equal symplectic balls was constructed in [KaTo2].

1.6.2. Seshadri constants. Our previous results can also be used to estimate Seshadri constants, which are interesting invariants in algebraic geometry. Recall that for a compact complex manifold (M,J) of complex dimension n and an ample line bundle $L \to M$, the **Seshadri constant** of L at a point $x \in M$ is defined as the nonnegative real number

$$\varepsilon(L,x) := \inf_{C \ni x} \frac{\int_C c_1(L)}{\mathrm{mult}_x C}, \tag{28}$$

where the infimum is taken over all irreducible holomorphic curves C passing through the point x, and $\mathrm{mult}_x C$ is the multiplicity of C at x ([De]). The global Seshadri constant is defined by

$$\varepsilon(L) := \inf_{x \in M} \varepsilon(L,x).$$

Seshadri's criterion for ampleness says that L is ample if and only if $\varepsilon(L) > 0$. The cohomology class $c_1(L)$ can be represented by a J-compatible Kähler form ω_L (the curvature form for a suitable metric connection on L). Denote $L^n = \int_M \omega_L^n = n!\,\mathrm{Vol}(M,\omega_L)$. Then $\varepsilon(L,x)$ has the elementary upper bound

$$\varepsilon(L,x) \le \sqrt[n]{L^n}. \tag{29}$$

Biran and Cieliebak [BiCi, Prop. 6.2.1] gave a better upper bound, i.e.,

$$\varepsilon(L) \le \mathcal{W}_G(M,\omega_L).$$

However, it is difficult to estimate $\mathcal{W}_G(M,\omega_L)$. Together with Theorem 1.27 we get

THEOREM 1.36: *For a closed connected complex manifold of complex dimension at least* 2,

$$\varepsilon(L) \le \mathrm{GW}(M,\omega_L).$$

Remark 1.37: By Definition 26, if $\mathrm{GW}(M,\omega_L)$ is finite then (M,ω_L) is symplectic uniruled. So Theorem 1.36 has only actual sense for uniruled (M,J). In this case our upper bound $\mathrm{GW}(M,\omega_L)$ is better than $\sqrt[n]{L^n}$ in (29). As an example, let us consider the hyperplane $[H]$ in $\mathbb{C}P^n$. It is ample, and the Fubini–Study form ω_{FS} with $\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = 1$ is a Kähler representative of $c_1([H])$. Let p_1 and p_2 denote the projections of the product $\mathbb{C}P^n \times \mathbb{C}P^n$ to the first and second factors. For an integer $m > 1$ the line bundle $p_1^*[H]+p_2^*(m[H]) \to \mathbb{C}P^n \times \mathbb{C}P^n$ is ample and $c_1(p_1^*[H]+p_2^*(m[H]))$ has a Kähler form representative $\omega_{\rm FS} \oplus m\omega_{\rm FS}$. From the proof of Theorem 1.16 it easily follows that

$$\mathrm{GW}(\mathbb{C}P^n \times \mathbb{C}P^n, \omega_{\rm FS} \oplus m\omega_{\rm FS}) \leq 1.$$

(In fact, equality holds.) But a direct computation gives

$$\sqrt[2n]{(p_1^*[H]+p_2^*(m[H]))^{2n}} = \left(\int_{\mathbb{C}P^n\times\mathbb{C}P^n}(\omega_{\rm FS}\oplus m\omega_{\rm FS})^{2n}\right)^{\frac{1}{2n}} \sqrt[2n]{m}\cdot \sqrt[2n]{\frac{(2n)!}{n!n!}} > 1.$$

From the above arguments and the subsequent proofs the reader can see that some of our results are probably not optimal. In fact, it is very possible that using our methods one can obtain better results in some cases ([Lu7] and [Lu9]). We content ourselves with illustrating the new ideas and methods.

The paper is organized as follows. In Section 2 we give the proofs of Lemma 1.4, Theorems 1.5, 1.8 and Proposition 1.7. The proof of Theorem 1.10 is given in Section 3. In Section 4 we prove Theorems 1.15, 1.16, 1.17 and 1.21. In Section 5 we prove Theorems 1.22 and 1.24. Theorem 1.35 is proved in Section 6. In the Appendix we discuss some related results on the Gromov–Witten invariants of product manifolds.

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2. Proofs of Lemma 1.4, Theorems 1.5 and 1.8 and Proposition 1.7

We first give two lemmas. They are key to our proofs in this section and the next one. According to Lemma 4.4 on page 107 and Exercise 9 on page 108 of [Hi] we have

LEMMA 2.1: *If N is a connected smooth manifold and $W \subset \mathrm{Int}(N)$ a compact smooth submanifold with connected boundary and of codimension zero, then ∂W separates N in the sense that $\mathrm{Int}(N) \setminus \partial W$ has exactly two connected components and the topological boundary of each component is ∂W. In this case ∂W has a neighborhood in N which is a product $\partial W \times (-2,2)$ with ∂W corresponding to $\partial W \times \{0\}$. If W is only contained in N then ∂W has a neighborhood in W which is a product $\partial W \times (-2,0]$.*

From Lemma 12.27 in [McSal] we easily derive

LEMMA 2.2: *Given a Riemannian metric g on M, there exists $\rho = \rho(g, M) > 0$ such that for every smooth function H on M with*

$$\sup_{x \in M} \|\nabla_g \nabla_g H(x)\|_g < \rho$$

the Hamiltonian equation $\dot{x} = X_H(x)$ has no nonconstant fast periodic solutions. In particular, the conclusion holds if $\|H\|_{C^2} < \rho$. Here ∇_g is the Levi-Civita connection of g and norms are taken with respect to g.

From Darboux's theorem we obtain

LEMMA 2.3: *Let (M, ω) be a $2n$-dimensional symplectic manifold, and $B^{2n}(r) = \{z \in \mathbb{R}^{2n} : |z| \le r\}$ with $r > 0$. Then for any $z_0 \in \mathrm{Int}(M)$ and any small $\varepsilon > 0$ there exist $r > 0$, a symplectic embedding φ: $(B^{2n}(2r), \omega_0) \to (M, \omega)$ with $\varphi(0) = z_0$ and a smooth function $H^{\varphi}_{r,\varepsilon}$: $M \to \mathbb{R}$ such that:*

(i) *$H^{\varphi}_{r,\varepsilon} = 0$ outside $\mathrm{Int}(\varphi(B^{2n}(2r))$, and $H^{\varphi}_{r,\varepsilon} = \varepsilon$ on $\varphi(B^{2n}(r))$.*

(ii) *$H^{\varphi}_{r,\varepsilon}$ is constant $h(s)$ along $\varphi(\{|z| = s\})$ for any $s \in [0, 2r]$, where h: $[0, 2r] \to [0, \varepsilon]$ is a nonnegative smooth function which is strictly decreasing on $[r, 2r]$. Consequently, $H^{\varphi}_{r,\varepsilon}(\varphi(z)) > H^{\varphi}_{r,\varepsilon}(\varphi(z'))$ if $r \le |z| < |z'| \le 2r$, and $H^{\varphi}_{r,\varepsilon}$ has no critical values in $(0, \varepsilon)$.*

(iii) *$\dot{x} = X_{H^{\varphi}_{r,\varepsilon}}(x)$ has no nonconstant fast periodic solutions.*

Proof of Lemma 1.4: **Case (i)**. We only need to prove that

$$C_{HZ}(M, \omega; pt, pt) \ge c_{HZ}(M, \omega).$$

To this end it suffices to construct for any $H \in \mathcal{H}_{ad}(M,\omega)$ an

$$F \in \mathcal{H}_{ad}(M,\omega; pt, pt)$$

such that $\max F \geq \max H$. By the definition there exist a nonempty open subset U and a compact subset $K \subset M \setminus \partial M$ such that: (a) $H|_U = 0$ and $H|_{M\setminus K} = \max H$, (b) $0 \leq H \leq \max H$, (c) $\dot{x} = X_H(x)$ has no nonconstant fast periodic solutions. These imply that $U \subset \mathrm{Int}(K)$. By the illustrations below the definition of c_{HZ} in §1.2 we may assume that $M \setminus K \neq \emptyset$. Then both U and $M \setminus K$ are nonempty open sets because M is a closed manifold. For a given small $\varepsilon > 0$ we may take symplectic embeddings φ and ψ from $(B^{2n}(2r), \omega_0)$ to (M,ω) such that

$$\varphi(B^{2n}(2r)) \subset U \quad \text{and} \quad \psi(B^{2n}(2r)) \subset M \setminus K.$$

Let $H^{\varphi}_{r,\varepsilon}$ and $H^{\psi}_{r,\varepsilon}$ be the corresponding functions as in Lemma 2.3. Since $H^{\varphi}_{r,\varepsilon}$ (resp. $H^{\psi}_{r,\varepsilon}$) is equal to zero outside $\varphi(B^{2n}(2r))$ (resp. $\psi(B^{2n}(2r))$) we can define a smooth function $\widetilde{H}: M \to \mathbb{R}$ by

$$\widetilde{H}(x) = \begin{cases} \max H + H^{\psi}_{r,\varepsilon}(x) & \text{if } x \in M \setminus K, \\ H(x) & \text{if } x \in K \setminus U, \\ -H^{\varphi}_{r,\varepsilon}(x) & \text{if } x \in U. \end{cases}$$

Define $F = \widetilde{H} + \varepsilon$. Then $\max F = \max H + 2\varepsilon \geq \max H$, $\min F = 0$ and $\dot{x} = X_F(x)$ has no nonconstant fast periodic orbits in M.

Since M is a closed manifold, $M \setminus \mathrm{Int}(\psi(B^{2n}(r)))$ is a compact submanifold with boundary $\psi(\partial B^{2n}(r))$. It follows that $F \in \mathcal{H}_{ad}(M,\omega; pt, pt)$ with $P(F) = \varphi(B^{2n}(r))$ and $Q(F) = M \setminus \mathrm{Int}(\psi(B^{2n}(r)))$. The desired result follows.

Going through the above proof we see that if $H \in \mathcal{H}^{\circ}_{ad}(M,\omega)$, i.e., $\dot{x} = X_H(x)$ has no nonconstant contractible fast periodic solutions, then $F \in \mathcal{H}^{\circ}_{ad}(M,\omega; pt, pt)$. This implies that $C^{\circ}_{HZ}(M,\omega) = c^{\circ}_{HZ}(M,\omega)$.

Case (ii). The arguments are similar. We only indicate different points. Let $H \in \mathcal{H}_{ad}(M,\omega)$. For a compact subset $K(H) \subset M \setminus \partial M$ we find by assumption a compact submanifold W with connected boundary and of codimension zero such that $K(H) \subset W$. Since $K(H)$ is compact and disjoint from ∂M we can assume that $K(H)$ is also disjoint from ∂W. By Lemma 2.1 we can choose embeddings

$$\Phi: [-5, 0] \times \partial W \to M$$

such that $\Phi(\{0\} \times \partial W) = \partial W$ and

$$\Phi([-5,0] \times \partial W) \subset W \quad \text{and} \quad K(H) \cap \Phi([-5,0] \times \partial W) = \emptyset.$$

For each $t \in [-5, 0]$ the set

$$W_t := W \setminus \Phi((t, 0] \times \partial W)$$

is a compact submanifold of M which is diffeomorphic to W. By shrinking $\varepsilon > 0$ in Case (i) if necessary, one easily constructs a smooth function H_ε: $M \to \mathbb{R}$ such that

(a) $H_\varepsilon = 0$ in $\mathrm{Int}(W_{-4})$ and $H_\varepsilon = \varepsilon$ outside W_{-1};
(b) $0 \le H_\varepsilon \le \varepsilon$ and each $c \in (0, \varepsilon)$ is a regular value of H_ε;
(c) H_ε is constant $f(s)$ along $\Phi(\{s\} \times \partial W)$ for any $s \in [-5, 0]$, where f: $[-5, 0] \to [0, \varepsilon]$ is a nonnegative smooth function which is strictly increasing in $[-4, -1]$;
(d) $\dot{x} = X_{H_\varepsilon}(x)$ has no nonconstant fast periodic solutions.

Let $H^{\varphi}_{r,\varepsilon}$ be as in Case (i). We can define a smooth function $\widetilde{H}$: $M \to \mathbb{R}$ by

$$\widetilde{H}(x) = \begin{cases} \max H + H_\varepsilon(x) & \text{if } x \in M \setminus K, \\ H(x) & \text{if } x \in K \setminus U, \\ -H^{\varphi}_{r,\varepsilon}(x) & \text{if } x \in U, \end{cases}$$

and set $F = \widetilde{H} + \varepsilon$. Then $\max F \ge \max H$, $\min F = 0$ and $\dot{x} = X_F(x)$ has no nonconstant fast periodic solutions. As in Case (i) one checks that $F \in \mathcal{H}_{ad}(M, \omega; pt, pt)$ with $P(F) = \varphi(B^{2n}(r))$ and $Q(F) = M \setminus \mathrm{Int}(W_{-1})$. So we have $\max H \le \max F \le C_{HZ}(M, \omega)$ for any $H \in \mathcal{H}_{ad}(M, \omega)$, and thus $c_{HZ}(M, \omega) \le C_{HZ}(M, \omega)$. As above we get that $C_{HZ}(M, \omega) = c_{HZ}(M, \omega)$ and $C^{\circ}_{HZ}(M, \omega) = c^{\circ}_{HZ}(M, \omega)$. ∎

Proof of Theorem 1.5: (i) We take $H \in \mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$. Let $P = P(H)$ and $Q = Q(H)$ be the corresponding submanifolds in Definition 1.2, and α_0, α_∞ the chain representatives. Define $G = -H + \max H$. Then $0 \le G \le \max G = \max H$, $G|_P = \max G$, $G|_{M \setminus \mathrm{Int}(Q)} = 0$ and $X_G = -X_H$. Therefore, $G \in \mathcal{H}_{ad}(M, \omega; \alpha_\infty, \alpha_0)$, and (i) follows.

(ii) is a special case of (v), and (iv) and (vi) are clear.

For (iii), note that $B^{2n}(1)$ and $Z^{2n}(1)$ are contractible. One can slightly modify the proofs of Lemma 3 and Theorem 2 in Chapter 3 of [HZ2] to show that $C^{(2)}_{HZ}(B^{2n}(1), \omega_0; \alpha_0, \alpha_\infty) \ge \pi$ and $C^{(2\circ)}_{HZ}(Z^{2n}(1), \omega_0; \alpha_0, \alpha_\infty) \le \pi$. Then (iii) follows from (v) and definitions:

$$\begin{aligned} \pi &\le C^{(2)}_{HZ}(B^{2n}(1), \omega_0; \alpha_0, \alpha_\infty) \\ &\le C^{(2\circ)}_{HZ}(B^{2n}(1), \omega_0; \alpha_0, \alpha_\infty) \\ &\le C^{(2\circ)}_{HZ}(Z^{2n}(1), \omega_0; \alpha_0, \alpha_\infty) \\ &\le \pi. \end{aligned}$$

For (v) we only prove the first claim. The second claim then follows together with the argument in [Lu1]. For $H \in \mathcal{H}_{ad}(M_1, \omega_1; \alpha_0, \alpha_\infty)$ let the submanifolds P_1 and Q_1 of (M_1, ω_1) be as in Definition 1.2. Set $P_2 = \psi(P_1)$ and $Q_2 = \psi(Q_1)$, and define $\psi_*(H) \in C^\infty(M_2, \mathbb{R})$ by

$$\psi_*(H)(x) = \begin{cases} H \circ \psi^{-1}(x) & \text{if } x \in \psi(M_1), \\ \max H & \text{if } x \notin \psi(M_1). \end{cases}$$

It is clear that $\psi_*(H) \in \mathcal{H}_{\rm ad}(M_2, \omega_2; \psi_*(\alpha_0), \psi_*(\alpha_\infty))$, and so (v) follows.

To prove (vii) we only need to show that $\mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ is nonempty under the assumptions there. Without loss of generality let α_0 be represented by a compact connected submanifold $S \subset \operatorname{Int} M$ without boundary. Since $\dim \alpha_0 + \dim \alpha_\infty \leq \dim M - 1$ it follows from intersection theory that there is a cycle representative $\tilde{\alpha}_\infty$ of α_∞ such that $S \cap \tilde{\alpha}_\infty = \emptyset$.

Choose a Riemannian metric g on M. For $\epsilon > 0$ let $\mathcal{N}_\epsilon$ be the closed ϵ-ball bundle in the normal bundle along S, and let $\exp\colon \mathcal{N}_\epsilon \to M$ be the exponential map. For $\epsilon > 0$ small enough, $P = S_\epsilon = \exp(\mathcal{N}_\epsilon)$ and $Q = S_{2\epsilon} = \exp(\mathcal{N}_{2\epsilon})$ are smooth compact submanifolds of M of codimension zero, and $S_{2\epsilon}$ is still disjoint from $\tilde{\alpha}_\infty$. Since $\dim S = \dim \alpha_0 \leq \dim M - 2$, both P and Q have connected boundary.

Take a smooth function $f\colon R \to \mathbb{R}$ such that $f(t) = 0$ for $t \leq \varepsilon^2$, $f(t) = 1$ for $t \geq 4\varepsilon^2$ and $f'(t) > 0$ for $\varepsilon^2 < t < 4\varepsilon^2$. We define a smooth function $F\colon M \to \mathbb{R}$ by $F(x) = 0$ for $x \in P$, $F(x) = 1$ for $x \in M \setminus Q$ and $F(x) = f(\|v_x\|_g^2)$ for $x = (s_x, v_x) \in S_{2\varepsilon}$. In view of Lemma 2.2 above, for $\delta > 0$ sufficiently small the function $F_\delta = \delta F$ belongs to $\mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$. ∎

Proof of Proposition 1.7: Note that every function H in $\mathcal{H}_{ad}(W, \omega; \tilde{\alpha}_0, pt)$ can be viewed as one in $\mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ in a natural way, and so (6) follows.

If the inclusion $W \hookrightarrow M$ induces an injective homomorphism $\pi_1(W) \to \pi_1(M)$ then each function H in $\mathcal{H}^\circ_{ad}(W, \omega; \tilde{\alpha}_0, pt)$ can be viewed as one in $\mathcal{H}^\circ_{ad}(M, \omega; \alpha_0, \alpha_\infty)$. Therefore we get (8).

To prove (10) let us take a function $H \in \mathcal{H}_{ad}(M \setminus W, \omega; \tilde{\alpha}_\infty, pt)$. Suppose that $P(H) \subset Q(H) \subset \operatorname{Int}(M \setminus W)$ are submanifolds associated with H. Then $H = \max H$ on $(M \setminus W) \setminus Q$. Therefore we can extend H to M by setting $H = \max H$ on W. We denote this extension by $\bar{H}$. Since we have assumed that α_0 has a cycle representative whose support is contained in $\operatorname{Int}(W) \subset M \setminus Q$, $\bar{H}$ belongs to $\mathcal{H}_{ad}(M, \omega; \alpha_\infty, \alpha_0)$.

If $H \in \mathcal{H}^\circ_{ad}(M \setminus W, \omega; \tilde{\alpha}_\infty, pt)$ and the inclusion $M \setminus W \hookrightarrow M$ induces an injective homomorphism $\pi_1(M \setminus W) \to \pi_1(M)$ then the above $\bar{H}$ belongs to $\mathcal{H}^\circ_{ad}(M, \omega; \alpha_\infty, \alpha_0)$. This implies (11).

For (12) we only need to prove that $\mathcal{W}_G(M,\omega) \le C_{HZ}^{(2)}(M,\omega;pt,\alpha)$ since $C_{HZ}^{(2)}(M,\omega;pt,\alpha) \le C_{HZ}^{(2\circ)}(M,\omega;pt,\alpha)$. For any given symplectic embedding ψ: $(B^{2n}(r),\omega_0) \to (\mathrm{Int}(M),\omega)$ and sufficiently small $\epsilon > 0$, we can choose a representative of α with support in $M\setminus\psi(B^{2n}(r-\epsilon))$ because $\dim\alpha \le \dim M-1$. By (5) and (7) we have

$$\pi(r-\epsilon)^2 = \mathcal{W}_G(\psi(B^{2n}(r-\epsilon)),\omega) \le C_{HZ}(\psi(B^{2n}(r-\epsilon)),\omega) \le C_{HZ}^{(2)}(M,\omega;pt,\alpha).$$

With $\epsilon \to 0$ we arrive at the desired conclusion. ∎

Proof of Theorem 1.8: To prove (13) let W and α_0,α_∞ satisfy the assumptions in Theorem 1.8. For $H \in \mathcal{H}_{ad}(W,\omega;\tilde{\alpha}_0,pt)$ and $G \in \mathcal{H}_{ad}(M\setminus W,\omega;\tilde{\alpha}_\infty,pt)$ let $P_1 \subset \mathrm{Int}(Q_1) \subset Q_1 \subset \mathrm{Int}(W)$ and $P_2 \subset \mathrm{Int}(Q_2) \subset Q_2 \subset M\setminus W$ be corresponding submanifolds as in Definition 1.2. Then $H|_{P_1} = 0$, $H|_{W\setminus\mathrm{Int}(Q_1)} = \max H$ and $G|_{P_2} = 0$, $G|_{(M\setminus W)\setminus\mathrm{Int}(Q_2)} = \max G$. Define K: $M \to \mathbb{R}$ by

$$K(x) = \begin{cases} H(x), & \text{if } x \in W,\\ \max H + \max G - G(x), & \text{if } x \in M\setminus W. \end{cases}$$

This is a smooth function and belongs to $\mathcal{H}_{ad}(M,\omega;\alpha_0,\alpha_\infty)$ with $P(K) = P_1$ and $Q(K) = M\setminus\mathrm{Int}(P_2)$. But $\max K = \max H + \max G$. This leads to (13). ∎

The following corollary of Theorem 1.8 will be useful later on.

COROLLARY 2.4: *Under the assumptions of Theorem 1.8, let (N,σ) be another closed connected symplectic manifold and $\beta \in H_*(N;\mathbb{Q})\setminus\{0\}$. Then*

$$\begin{aligned}&C_{HZ}^{(2)}(N\times W,\sigma\oplus\omega;\beta\times\tilde{\alpha}_0,pt) + C_{HZ}^{(2)}(N\times(M\setminus W),\sigma\oplus\omega;\beta\times\tilde{\alpha}_\infty,pt)\\ &\le C_{HZ}^{(2)}(N\times M,\sigma\oplus\omega;\beta\times\alpha_0,\beta\times\alpha_\infty),\end{aligned}$$

and

$$\begin{aligned}&C_{HZ}^{(2\circ)}(N\times W,\sigma\oplus\omega;\beta\times\tilde{\alpha}_0,pt) + C_{HZ}^{(2\circ)}(N\times(M\setminus W),\sigma\oplus\omega;\beta\times\tilde{\alpha}_\infty,pt)\\ &\le C_{HZ}^{(2\circ)}(N\times M,\sigma\oplus\omega;\beta\times\alpha_0,\beta\times\alpha_\infty)\end{aligned}$$

if both inclusions $W \hookrightarrow M$ and $M\setminus W \hookrightarrow M$ also induce an injective homomorphisms $\pi_1(W) \to \pi_1(M)$ and $\pi_1(M\setminus W) \to \pi_1(M)$.

3. Proof of Theorem 1.10

We wish to reduce the proof of this theorem to the arguments in [LiuT]. Liu–Tian's approach is to introduce the Morse theoretical version of Gromov–Witten invariants. In their work the paper [FHS] plays an important role. To show how the arguments in [LiuT] apply to our case we need to recall some related material from [FHS].

Consider the vector space $\mathcal{S} = \{S \in \mathbb{R}^{2n\times 2n} \mid S^T = S\}$ of symmetric $(2n\times 2n)$-matrices. It has an important subset $\mathcal{S}^{2n}_{\rm reg}$ consisting of all matrices $S \in \mathcal{S}$ such that for any four real numbers a, b, α, β the system of equations

$$\begin{cases} (SJ_0 - J_0S - aI_{2n} - bJ_0)\zeta = 0 \\ (SJ_0 - J_0S - aI_{2n} - bJ_0)S\zeta - \alpha\zeta - \beta J_0\zeta = 0 \end{cases} \tag{30}$$

has no nonzero solution $\zeta \in \mathbb{R}^{2n\times 2n}$, where I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$ and

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

It has been proved in Theorem 6.1 of [FHS] that for $n \ge 2$ the set $\mathcal{S}^{2n}_{\rm reg}$ is open and dense in $\mathcal{S}$ and $\tau\Phi^T S\Phi \in \mathcal{S}^{2n}_{\rm reg}$ for any $S \in \mathcal{S}^{2n}_{\rm reg}$, any $\Phi \in GL(n,\mathbb{C}) \cap O(2n)$ and any real number $\tau \ne 0$. In view of Definition 7.1 in [FHS] and the arguments in [McSl] we introduce

Definition 3.1: A nondegenerate critical point p of a smooth function H on a symplectic manifold (M,ω) is called **strong admissible** if it satisfies the following two conditions:

(i) the spectrum of the linear transformation $DX_H(p)\colon T_pM \to T_pM$ is contained in $\mathbb{C} \setminus \{\lambda i \mid 2\pi \le \pm\lambda < +\infty\}$;

(ii) there exists $J_p \in \mathcal{J}(T_pM, \omega_p)$ such that for some (and hence every) unitary frame $\Phi\colon \mathbb{R}^{2n} \to T_pM$ (i.e., $\Phi J_0 = J_p\Phi$ and $\Phi^*\omega_p = \omega_0$) we have

$$S = J_0\Phi^{-1}DX_H(p)\Phi \in \mathcal{S}^{2n}_{\rm reg}.$$

Definition 3.2: An $(\alpha_0, \alpha_\infty)$-admissible (resp. $(\alpha_0,\alpha_\infty)^\circ$-admissible) function H in Definition 1.2 is said to be **(α_0,α_∞)-strong admissible** (resp. **$(\alpha_0,\alpha_\infty)^\circ$-strong admissible**) if instead of condition (5) it satisfies the stronger condition

(5′) H has only finitely many critical points in $\mathrm{Int}(Q) \setminus P$, and each of them is strong admissible in the sense of Definition 3.1.

Let us respectively denote by

$$\mathcal{H}_{sad}(M,\omega;\alpha_0,\alpha_\infty) \quad \text{and} \quad \mathcal{H}^\circ_{sad}(M,\omega;\alpha_0,\alpha_\infty) \tag{31}$$

the set of $(\alpha_0, \alpha_\infty)$-strong admissible and $(\alpha_0, \alpha_\infty)^\circ$-strong admissible functions. They are subsets of $\mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ and $\mathcal{H}^\circ_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ respectively. The following lemma is key to our proof.

LEMMA 3.3: *If* $\dim M \geq 4$, *then* $\mathcal{H}_{sad}(M, \omega; \alpha_0, \alpha_\infty)$ (*resp.* $\mathcal{H}^\circ_{sad}(M, \omega; \alpha_0, \alpha_\infty)$) *is* C^0*-dense in* $\mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ (*resp.* $\mathcal{H}^\circ_{ad}(M, \omega; \alpha_0, \alpha_\infty)$).

Proof: Let $F \in \mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ (resp. $\mathcal{H}^\circ_{ad}(M, \omega; \alpha_0, \alpha_\infty)$). We shall prove that for any small $\epsilon > 0$ there exists a $G \in \mathcal{H}_{sad}(M, \omega; \alpha_0, \alpha_\infty)$ (resp. $\mathcal{H}^\circ_{sad}(M, \omega; \alpha_0, \alpha_\infty)$) such that

$$\max F \geq \max G \geq \max F - \epsilon. \tag{32}$$

Our proof is inspired by the proof of Proposition 3.1 in [Schl].

Let C_F (resp. c_F) be the largest (resp. smallest) critical value of F in $(0, \max F)$. If there are no such critical values, there is nothing to show. If $c_F = C_F$, then it is the only critical value of F in $(0, \max F)$, and this case can easily be proved by the following method. So we now assume $c_F \leq C_F$. Then by Definition 1.2(5) we have

$$0 < c_F < C_F < \max F.$$

Let $C(F)$ be the set of critical values of F. It is compact and has zero Lebesgue measure, so that for small $\epsilon > 0$ we can choose regular values of F,

$$b'_0 < a'_1 < b'_1 < \cdots < a'_{k-1} < b'_{k-1} < a'_k,$$

such that:

(i) $0 < b'_0 < c_F$ and $C_F < a'_k < \max F$.

(ii) $[a'_i, b'_i] \subset [c_F, C_F] \setminus C(F)$, $i = 1, \ldots, k-1$.

(iii) $\sum_{i=1}^{k-1}(b'_i - a'_i) + b'_0 + \max F - a'_k > \max F - \epsilon$.

Furthermore, we may also take regular values of F,

$$b_0 < a_1 < b_1 < \cdots < a_{k-1} < b_{k-1} < a_k,$$

such that

$$b_0 < b'_0, \quad a_k > a'_k, \quad a'_i < a_i < b_i < b'_i, \quad i = 1, \ldots, k-1,$$
$$\sum_{i=1}^{k-1}(b_i - a_i) + b_0 + \max F - a_k > \max F - 2\epsilon.$$

Consider the piecewise-linear function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(t) = \begin{cases} t & \text{for } t \le b_0, \\ b_0 & \text{for } b_0 \le t \le a_1, \\ t - a_1 + b_0 & \text{for } a_1 \le t \le b_1, \\ b_1 - a_1 + b_0 & \text{for } b_1 \le t \le a_2, \\ t - a_2 + (b_1 - a_1) + b_0 & \text{for } a_2 \le t \le b_2, \\ \cdots & \text{for } \cdots, \\ t - a_{k-1} + \sum_{i=1}^{k-2}(b_i - a_i) + b_0 & \text{for } a_{k-1} \le t \le b_{k-1}, \\ \sum_{i=1}^{k-1}(b_i - a_i) + b_0 & \text{for } b_{k-1} \le t \le a_k, \\ t - a_k + \sum_{i=1}^{k-1}(b_i - a_i) + b_0 & \text{for } t \ge a_k. \end{cases}$$

Then $\min\{f(t)|\ t \in [0, \max F]\} = 0$ and

$$\max\{f(t)|\ t \in [0, \max F]\} = \max F - a_k + \sum_{i=1}^{k-1}(b_i - a_i) + b_0 > \max F - 2\epsilon.$$

Note that $b_0 < a_1 < b_1 < \cdots < a_{k-1} < b_{k-1} < a_k$ are all nonsmooth points of f in $(0, \max F)$. By suitably smoothing f near these points we can get a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ satisfying:

$(h)_1$ $0 \le h'(t) \le 1$ for $t \in \mathbb{R}$;

$(h)_2$ $0 < h'(t) \le 1$ for $t \in [0, b'_0) \cup (a'_k, \max F] \cup (\bigcup_{i=1}^{k-1}(a'_i, b'_i))$;

$(h)_3$ $h(t) = f(t)$ for $t \in \bigcup_{i=0}^{k-1}[b'_i, a'_{i+1}]$;

$(h)_4$ $h(t) = f(t)$ near $t = 0$ and $t = \max F$.

Set $H = h \circ F$. Then $(h)_1$ and $(h)_4$ imply that $H \in \mathcal{H}_{ad}(M, \omega; \alpha_0, \alpha_\infty)$ and

$$\max H = h(\max F) = \max F - a_k + \sum_{i=1}^{k-1}(b_i - a_i) + b_0 > \max F - 2\epsilon. \tag{33}$$

Furthermore, one easily checks that

$(H)_1$ The critical values of H in $(0, \max H)$ are exactly b_0, $\sum_{i=1}^{j}(b_i - a_i) + b_0$, $j = 1, \ldots, k-1$;

$(H)_2$ The corresponding critical sets are respectively $\{b'_0 \le F \le a'_1\}$ and $\{b'_j \le F \le a'_{j+1}\}$, $j = 1, \ldots, k-1$;

$(H)_3$ $H = b_0$ on $\{b'_0 \le F \le a'_1\}$;

$(H)_4$ $H = \sum_{i=1}^{j}(b_i - a_i) + b_0$ on $\{b'_j \le F \le a'_{j+1}\}$, $j = 1, \ldots, k-1$.

For each $0 \le s < \frac{1}{2}\min\{a'_{i+1} - b'_i, b'_i - a'_i, b'_0, \max F - a'_k|\ 0 \le i \le k-1\}$ we set

$$N_s := \bigcup_{i=0}^{k-1}\{b'_i - s \le F \le a'_{i+1} + s\}.$$

Since the set of regular values of F is open, both N_0 and N_s with sufficiently small $s>0$ are compact smooth submanifolds with boundary. For any open neighborhood $\mathcal{O}$ of N_0 we have also $N_s\subset\mathcal{O}$ if $s>0$ is small enough. By $(H)_3$ and $(H)_4$, $\nabla_g\nabla_g H=0$ on N_0 and thus we can choose

$$0<\delta<\frac{1}{4}\min\{a'_{i+1}-b'_i, b'_i-a'_i, b'_0, \max F-a'_k \mid 0\le i\le k-1\}$$

so small that

$$\sup_{x\in N_{2\delta}}\|\nabla_g\nabla_g H(x)\|_g<\rho/2.$$

Here ρ is given by Lemma 2.2. Let us take a smooth function $L\colon M\to\mathbb{R}$ such that

$(L)_1$ $\operatorname{supp}(L)\subset N_\delta$;

$(L)_2$ $\|L\|_{C^2}<\rho/2$ (and thus $\sup_{x\in N_{2\delta}}\|\nabla_g\nabla_g(H+L)(x)\|_g<\rho$);

$(L)_3$ $h(b'_i-2\delta)<H(x)+L(x)<h(a'_{i+1}+2\delta)$ for $x\in\{b'_i-\delta\le F\le a'_{i+1}+\delta\}$, $i=0,\dots,k-1$;

$(L)_4$ $H+L$ has only finitely many critical points in N_δ and each of them is strong admissible.

The condition $(L)_4$ can be assured by Lemma 7.2 (i) in [FHS]. To see that $(L)_3$ can be satisfied, note that $(h)_1$ implies that $h(b'_i-\delta)\le H(x)\le h(a'_{i+1}+\delta)$ as $b'_i-\delta\le F(x)\le a'_{i+1}+\delta$. By the choice of δ we have $b'_0-2\delta>0$, $a'_k+2\delta<\max F$ and

$$a'_i+\delta, a'_i+2\delta, b'_i-2\delta, b'_i-\delta\in(a'_i,b'_i),\quad i=1,\dots,k-1.$$

It follows from $(h)_2$ that for $i=0,\dots,k-1$,

$$h(b'_i-2\delta)<h(b'_i-\delta)<h(b'_i)\le h(a'_{i+1})<h(a'_{i+1}+\delta)<h(a'_{i+1}+2\delta). \tag{34}$$

Using these and $(L)_2$ we can easily choose L satisfying $(L)_3$. Set $G=H+L$. Then $P(G)=P(H)$, $Q(G)=Q(H)$ and

$$\max G=\max H\quad\text{and}\quad \min G=\min H=0. \tag{35}$$

Now we are in position to prove

$$G\in\mathcal{H}_{sad}(M,\omega;\alpha_0,\alpha_\infty)\quad(\text{resp. }\mathcal{H}^\circ_{sad}(M,\omega;\alpha_0,\alpha_\infty)).$$

First, the above construction shows that all critical values of G in $(0,\max G)$ sit in

$$\bigcup_{i=0}^{k-1}(h(b'_i-2\delta), h(a'_{i+1}+2\delta))$$

and the corresponding critical points sit in N_δ. It follows that G has only finitely many critical points in $\mathrm{Int}(Q) \setminus P$ and each of them is strong admissible.

Next we prove that X_G has no nonconstant fast periodic orbits. Assume that γ is such an orbit. It cannot completely sit in $M \setminus N_\delta$ because $G = H$ in $M \setminus N_\delta$. Moreover, Lemma 2.2 and $(L)_2$ imply that γ cannot completely sit in $N_{2\delta}$. So there must exist two points $\gamma(t_1)$ and $\gamma(t_2)$ such that $\gamma(t_1) \in \partial N_\delta$ and $\gamma(t_2) \in \partial N_{2\delta}$. Note that all possible values G takes on ∂N_δ (resp. $\partial N_{2\delta}$) are

$$h(b'_i - \delta), h(a'_{i+1} + \delta), \quad i = 0, \dots, k-1$$
$$(\text{resp. } h(b'_i - 2\delta), h(a'_{i+1} + 2\delta), \quad i = 0, \dots, k-1).$$

By (34) any two of them are different. But $G(\gamma(t_1)) = G(\gamma(t_2))$. This contradiction shows that X_G has no nonconstant fast periodic orbit. Clearly, this argument also implies that X_G has no nonconstant contractible fast periodic orbit if $F \in \mathcal{H}^\circ_{ad}(M, \omega; \alpha_0, \alpha_\infty)$.

Finally, (33) and (35) together give

$$\max G \geq \max F - 2\epsilon.$$

The desired conclusion is proved. ∎

As direct consequences of Lemma 3.3 and (2) we have

$$(36) \qquad \begin{cases} C^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) = \sup\{\max H | H \in \mathcal{H}_{sad}(M, \omega; \alpha_0, \alpha_\infty)\}, \\ C^{(2\circ)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) = \sup\{\max H |\ H \in \mathcal{H}^\circ_{sad}(M, \omega; \alpha_0, \alpha_\infty)\}. \end{cases}$$

Proof of Theorem 1.10: We only prove (15). The proof of (16) is similar. Without loss of generality we assume that $C^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) > 0$ and $\mathrm{GW}(M, \omega; \alpha_0, \alpha_\infty) < +\infty$. We need to prove that if

$$(37) \qquad \Psi_{A,g,m+2}(C; \alpha_0, \alpha_\infty, \beta_1, \dots, \beta_m) \neq 0$$

for homology classes $A \in H_2(M;\mathbb{Z})$, $C \in H_*(\overline{\mathcal{M}}_{g,m+2};\mathbb{Q})$ and $\beta_1, \dots, \beta_m \in H_*(M;\mathbb{Q})$ and integers $m \geq 1$ and $g \geq 0$, then

$$(38) \qquad C^{(2)}_{HZ}(M, \omega; \alpha_0, \alpha_\infty) \leq \omega(A).$$

Arguing by contradiction, we may assume by (36) that there exists $H \in \mathcal{H}_{sad}(M, \omega; \alpha_0, \alpha_\infty)$ such that $\max H > \omega(A)$. Then we take $\eta > 0$ such that

$$(39) \qquad \max H - 2\eta > \omega(A).$$

By the properties of H there exist two smooth compact submanifolds $P, Q \subset M$ with connected boundary and of codimension zero such that the conditions (1), (2), (3), (4), (6) in Definition 1.2 and (5′) in Definition 3.2 are satisfied. Changing H slightly near $\{H = 0\}$ and near $\{H = \max H\}$ in the class $\mathcal{H}_{sad}(M, \omega; \alpha_0, \alpha_\infty)$ and using Lemma 2.1, we can choose embeddings

$$\Phi\colon [-2, 0] \times \partial Q \to Q \setminus \mathrm{Int}(P) \quad \text{and} \quad \Psi\colon [0, 2] \times \partial P \to Q \setminus \mathrm{Int}(P)$$

such that:

(i) $\Phi(\{0\} \times \partial Q) = \partial Q$ and $\Psi(\{0\} \times \partial P) = \partial P$;

(ii) $\Phi([-2, 0] \times \partial Q) \cap \Psi([0, 2] \times \partial P) = \emptyset$;

(iii) H has no critical points in $\Phi([-2, 0) \times \partial Q) \cup \Psi((0, 2] \times \partial P)$ and is constant m_s on $\Phi(\{s\} \times \partial Q)$ and n_t on $\Psi(\{t\} \times \partial P)$ for each $s \in [-2, 0]$ and $t \in [0, 2]$;

(iv) $H(x) < m_s$ for any $x \in M \setminus \widehat{Q}_s$ and $s \in [-2, 0]$, and $n_t < H(x)$ for any $x \in M \setminus \widehat{P}_t$ and $t \in [0, 2]$, where

$$\widehat{Q}_s = (M \setminus Q) \cup \Phi([s, 0] \times \partial Q) \quad \text{and} \quad \widehat{P}_t = P \cup \Psi([0, t] \times \partial P).$$

Notice that the above assumptions imply

$$m_s < m_{s'} < \max H \quad \text{and} \quad 0 < n_t < n_{t'}$$

for $-2 < s < s' < 0$ and $0 < t < t' < 2$. Moreover, $\widehat{Q}_s$ (resp. $\widehat{P}_t$) is a smooth compact submanifold of M with boundary $\Phi(\{s\} \times \partial Q)$ (resp. $\Psi(\{t\} \times \partial P)$). Clearly, $\widehat{Q}_s \cap \widehat{P}_t = \emptyset$. For $\tau \in [0, 2]$ we abbreviate

$$B_\tau = \widehat{P}_\tau \cup \widehat{Q}_{-\tau}.$$

By the properties of H and (39) we find $\delta \in (0, 1)$ such that

$$m_{-2\delta} > \max H - \eta, \quad n_{2\delta} < \eta \quad \text{and} \quad \sup_{x \in B_{2\delta}} \|\nabla_g \nabla_g H(x)\|_g < \rho/2, \tag{40}$$

where ρ is as in Lemma 2.2. As before we may choose a smooth function $L\colon M \to \mathbb{R}$ such that

(a) $\mathrm{supp}(L) \subset \mathrm{Int}(B_\delta)$;

(b) $\|L\|_{C^2} < \min\{\rho/2, \eta\}$ (and thus $\sup_{x \in B_{2\delta}} \|\nabla_g \nabla_g (H + L)(x)\|_g < \rho$);

(c) $H + L$ has only finitely many critical points in $\mathrm{Int}(B_\delta)$, and each of them is also strong admissible;

(d) $m_{-2\delta} < H(x) + L(x)$ for $x \in \mathrm{Int}(\widehat{Q}_{-\delta})$;

(e) $H(x) + L(x) < n_{2\delta}$ for $x \in \mathrm{Int}(\widehat{P}_\delta)$.

As above, condition (c) is assured by Lemma 7.2 (i) in [FHS]. Set $F = H + L$. If $x \in B_\delta$ then either $F(x) > m_{-2\delta}$ or $F(x) < n_{2\delta}$. On the other hand, the above (a) and (iv) imply that $n_{2\delta} < F(x) < m_{-2\delta}$ if $x \in M \setminus B_{2\delta}$. This means that a solution of $\dot{x} = X_F(x)$ cannot go to B_δ from $M \setminus B_{2\delta}$ because F is constant along any solution of $\dot{x} = X_F(x)$. So any nonconstant solution of $\dot{x} = X_F(x)$ lies either in $B_{2\delta}$ or in $M \setminus B_\delta$. It follows from (a) and (b) that $\dot{x} = X_F(x)$ has no nonconstant fast periodic solutions. Using (40) and (a)–(e) again we get that F is a smooth Morse function on M satisfying

$(F)_1$ each critical point of F is strong admissible;

$(F)_2$ $\lambda \cdot F$ has no nontrivial periodic solution of period 1 for any $\lambda \in (0, 1]$;

$(F)_3$ $F(x) > \max H - \eta$ for $x \in \widehat{Q}_{-\delta}$, and $F(x) < \eta$ for any $x \in \widehat{P}_\delta$;

$(F)_4$ $\max F \leq \max H + \eta$ and $\min F \geq -\eta$.

As a consequence of $(F)_1$ we get that $\mathcal{J}_{ad}(M, \omega, X_F)$ is nonempty. From Lemma 7.2(iii) in [FHS] we also know that $\mathcal{J}_{ad}(M, \omega, X_F)$ is open in $\mathcal{J}(M, \omega)$ with respect to the C^0-topology. Therefore, we may choose a regular $J \in \mathcal{J}_{ad}(M, \omega, X_F)$ and then repeat the arguments in [LiuT] to define the Morse theoretical Gromov–Witten invariants

$$\Psi_{A,J_\lambda,\lambda F,g,m+2}(C; \alpha_0, \alpha_\infty, \beta_1, \ldots, \beta_m)$$

and to prove

$$\Psi_{A,J_\lambda,\lambda F,g,m+2}(C; \alpha_0, \alpha_\infty, \beta_1, \ldots, \beta_m) \equiv \Psi_{A,g,m+2}(C; \alpha_0, \alpha_\infty, \beta_1, \ldots, \beta_m) \tag{41}$$

for each $\lambda \in [0, 1]$. As in Lemma 7.2 of [LiuT] we can prove the corresponding moduli space $\mathcal{FM}(c_0, c_\infty; J_1, F, A)$ to be empty for any critical points $c_0 \in \widehat{P}_\delta$ and $c_\infty \in \widehat{Q}_{-\delta}$ of F. In fact, otherwise we may choose an element f in it. Then one easily gets the estimate

$$0 \leq E(f) = F(c_0) - F(c_\infty) + \omega(A). \tag{42}$$

(Note: From the proof of Lemma 7.2 in [LiuT] one may easily see that the energy identity above their Lemma 3.2 should read $E(f) = \omega(A) + H(c_-) - H(c_+)$.) From the above $(F)_3$ and (42) it follows that

$$\max H - 2\eta < F(c_\infty) - F(c_0) \leq \omega(A).$$

This contradicts (39). So $\mathcal{FM}(c_0, c_\infty; J_1, F, A)$ is empty and thus

$$\Psi_{A,J_1,F,g,m+2}(C; \alpha_0, \alpha_\infty, \beta_1, \ldots, \beta_m) = 0.$$

By (41) we get $\Psi_{A,g,m+2}(C;\alpha_0,\alpha_\infty,\beta_1,\dots,\beta_m) = 0$. This contradicts (37). (38) is proved. ∎

4. Proofs of Theorems 1.15, 1.16, 1.17 and 1.21

Proof of Theorem 1.15: We start with the matrix definition of the Grassmannian manifold $G(k,n) = G(k,n;\mathbb{C})$. Let $n = k+m$, $M(k,n;\mathbb{C}) = \{A \in \mathbb{C}^{k\times n}|\ \operatorname{rank} A = k\}$ and $\mathrm{GL}(k;\mathbb{C}) = \{Q \in \mathbb{C}^{k\times k}|\ \det Q \neq 0\}$. Then $\mathrm{GL}(k;\mathbb{C})$ acts freely on $M(k,n;\mathbb{C})$ from the left by matrix multiplication. The quotient $M(k,n;\mathbb{C})/\mathrm{GL}(k;\mathbb{C})$ is exactly $G(k,n)$. For $A \in M(k,n;\mathbb{C})$ we denote by $[A] \in G(k,n)$ the $\mathrm{GL}(k;\mathbb{C})$-orbit of A in $M(k,n;\mathbb{C})$, and by

$$\mathrm{Pr}\colon M(k,n;\mathbb{C}) \to G(k,n), \quad A \mapsto [A]$$

the quotient projection. Any representative matrix B of $[A]$ is called a **homogeneous coordinate** of the point $[A]$. For increasing integers $1 \le \alpha_1 < \cdots < \alpha_k \le n$ let $\{\alpha_{k+1},\dots,\alpha_n\}$ be the complement of $\{\alpha_1,\dots,\alpha_k\}$ in the set $\{1,2,\dots,n\}$. Let us write $A \in M(k,n;\mathbb{C})$ as $A = (A_1,\dots,A_n)$ and

$$A_{\alpha_1\cdots\alpha_k} = (A_{\alpha_1},\dots,A_{\alpha_k}) \in \mathbb{C}^{k\times k} \text{ and } A_{\alpha_{k+1}\cdots\alpha_n} = (A_{\alpha_{k+1}},\dots,A_{\alpha_n}) \in \mathbb{C}^{k\times m},$$

where $A_1,\dots,A_n$ are $k\times 1$ matrices. Define a subset of $M(k,n;\mathbb{C})$ by

$$V(\alpha_1,\dots,\alpha_k) = \{A \in M(k,n;\mathbb{C})|\ \det A_{\alpha_1\cdots\alpha_k} \neq 0\}$$

and set $U(\alpha_1,\dots,\alpha_k) = \mathrm{Pr}(V(\alpha_1,\dots,\alpha_k))$ and

$$\begin{gathered}\Theta(\alpha_1,\dots,\alpha_k)\colon U(\alpha_1,\dots,\alpha_k) \to \mathbb{C}^{k\times m} \equiv \mathbb{C}^{km},\\ [A] \to Z = (A_{\alpha_1\cdots\alpha_k})^{-1}A_{\alpha_{k+1}\cdots\alpha_n}.\end{gathered}$$

It is easily checked that this is a homeomorphism. Z is called the **local coordinate** of $[A] \in G(k,n)$ in the canonical coordinate neighborhood $U(\alpha_1,\dots,\alpha_k)$. Note that for any $Z \in \mathbb{C}^{k\times m}$ there must exist an $n\times n$ permutation matrix $P(\alpha_1,\dots,\alpha_k)$ such that for the matrix $A = (I^{(k)},Z)P(\alpha_1,\dots,\alpha_k)$ we have

$$A_{\alpha_1\cdots\alpha_k} = I^{(k)} \quad\text{and}\quad A_{\alpha_{k+1}\cdots\alpha_n} = Z. \tag{43}$$

Hereafter $I^{(k)}$ denotes the unit $k\times k$ matrix. It follows from this fact that for another set of increasing integers $1 \le \beta_1 < \cdots < \beta_k \le n$ the transition function $\Theta(\beta_1,\dots,\beta_k)\circ\Theta(\alpha_1,\dots,\alpha_k)^{-1}$ from $\Theta(\alpha_1,\dots,\alpha_k)(U(\alpha_1,\dots,\alpha_k))$ to $\Theta(\beta_1,\dots,\beta_k)(U(\beta_1,\dots,\beta_k))$ is given by

$$Z \to W = (W_{\beta_1\cdots\beta_k})^{-1}W_{\beta_{k+1}\cdots\beta_n},$$

where $(W_{\beta_1\cdots\beta_k}, W_{\beta_{k+1}\cdots\beta_n}) = (I,Z)P(\alpha_1,\dots,\alpha_k)P'(\beta_1,\dots,\beta_k)$. It is not hard to check that this transformation is biholomorphic. Thus

$$\{(U(\alpha_1,\dots,\alpha_k),\Theta(\alpha_1,\dots,\alpha_k))|1\le\alpha_1<\cdots<\alpha_k\le n\} \tag{44}$$

gives an atlas of the natural complex structure on $G(k,n)$, which is called the **canonical atlas**. It is not hard to prove that the canonical Kähler form $\sigma^{(k,n)}$ on $G(k,n)$ in such coordinate charts is given by

$$\frac{\sqrt{-1}}{2}\operatorname{tr}[(I^{(k)}+Z\overline{Z}')^{-1}dZ\wedge(I^{(m)}+\overline{Z}'Z)^{-1}d\overline{Z}'] = \frac{\sqrt{-1}}{2}\partial\bar\partial\log\det(I^{(k)}+Z\overline{Z}'),$$

where $dZ=(dz_{ij})_{1\le i\le k,1\le j\le m}$ and ∂, $\bar\partial$ are the differentials with respect to the holomorphic and antiholomorphic coordinates respectively (cf. [L]).

On the other hand, it is easy to see that

$$\begin{aligned}\tau_{k,n} &= \frac{\sqrt{-1}}{2}\partial\bar\partial\log\det(A\overline{A}')\\ &= \frac{\sqrt{-1}}{2}\operatorname{tr}[-(A\overline{A}')^{-1}dA\wedge\overline{A}'(A\overline{A}')^{-1}Ad\overline{A}'+(A\overline{A}')^{-1}dA\wedge d\overline{A}']\end{aligned}$$

is an invariant Kähler form on $M(k,n;\mathbb{C})$ under the left action of $\mathrm{GL}(k;\mathbb{C})$. Thus it descends to a symplectic form $\widehat{\tau}_{k,n}$ on $G(k,n;\mathbb{C})$. If $A=(I^{(k)},Z)$ it is easily checked that

$$\begin{aligned}&\frac{\sqrt{-1}}{2}\operatorname{tr}[-(A\overline{A}')^{-1}dA\wedge\overline{A}'(A\overline{A}')^{-1}Ad\overline{A}'+(A\overline{A}')^{-1}dA\wedge d\overline{A}']\\ =&\frac{\sqrt{-1}}{2}\operatorname{tr}[-(I^{(k)}+Z\overline{Z}')^{-1}dZ\wedge\overline{Z}'(I^{(k)}+Z\overline{Z}')^{-1}Zd\overline{Z}'\\ &\qquad+(I^{(k)}+Z\overline{Z}')^{-1}dZ\wedge d\overline{Z}']\\ =&\frac{\sqrt{-1}}{2}\operatorname{tr}[(I^{(k)}+Z\overline{Z}')^{-1}dZ\wedge(I^{(m)}+\overline{Z}'Z)^{-1}d\overline{Z}'].\end{aligned}$$

It follows that $\widehat{\tau}_{k,n}=\sigma^{(k,n)}$. Since $\mathrm{Pr}^*\,\widehat{\tau}_{k,n}=\tau_{k,n}$ we arrive at

$$\mathrm{Pr}^*\,\sigma^{(k,n)}=\tau_{k,n}. \tag{45}$$

As usual, if we identify $z=(z_{11},\dots,z_{1m},z_{21},\dots,z_{2m},\dots,z_{k1},\dots,z_{km})\in\mathbb{C}^{km}$ with the matrix $Z=(z_{ij})_{1\le i\le k,1\le j\le m}$ the standard symplectic form in $\mathbb{C}^{km}$ becomes

$$\omega^{(km)}=\frac{\sqrt{-1}}{2}\operatorname{tr}[dZ\wedge d\overline{Z}'].$$

Denote

$$M^0(k,n;\mathbb{C}) = \{A \in M(k,n;\mathbb{C}) |\ A\overline{A}' = I^{(k)}\}.$$

Then

$$\tau_{k,n}|_{M^0(k,n;\mathbb{C})} = \omega^{(km)}|_{M^0(k,n;\mathbb{C})}. \tag{46}$$

In fact, since $A\overline{A}' = I^{(k)}$ we have that $dA\overline{A}' + Ad\overline{A}' = 0$ and thus

$$\begin{aligned}&\frac{\sqrt{-1}}{2}\operatorname{tr}[-(A\overline{A}')^{-1}dA \wedge \overline{A}'(A\overline{A}')^{-1}Ad\overline{A}' + (A\overline{A}')^{-1}dA \wedge d\overline{A}']\\ =&\frac{\sqrt{-1}}{2}\operatorname{tr}[dA \wedge \overline{d}A'] + \frac{\sqrt{-1}}{2}\operatorname{tr}[dA\overline{A}' \wedge dA\overline{A}'].\end{aligned}$$

We want to prove the second term is zero. A direct computation yields

$$\begin{aligned}\operatorname{tr}[dA\overline{A}' \wedge dA\overline{A}'] &= \sum_{i=1}^{k}\sum_{j=1}^{k}(\sum_{s=1}^{n}\bar{a}_{js}da_{is}) \wedge (\sum_{s=1}^{n}\bar{a}_{is}da_{js})\\ &= \sum_{j=1}^{k}\sum_{i=1}^{k}(\sum_{s=1}^{n}\bar{a}_{is}da_{js}) \wedge (\sum_{s=1}^{n}\bar{a}_{js}da_{is}) \quad \text{(interchanging } i,j)\\ &= -\sum_{i=1}^{k}\sum_{j=1}^{k}(\sum_{s=1}^{n}\bar{a}_{js}da_{is}) \wedge (\sum_{s=1}^{n}\bar{a}_{is}da_{js}).\end{aligned}$$

Hence $\operatorname{tr}[dA\overline{A}' \wedge dA\overline{A}'] = 0$. (46) is proved.

LEMMA 4.1: *For the classical domain of the first type (cf. [L])*

$$R_I(k,m) = \{Z \in \mathbb{C}^{k\times m} |\ I^{(k)} - Z\overline{Z}' > 0\},$$

the map

$$\Phi\colon (R_I(k,m), \omega^{(km)}) \to (\mathbb{C}^{k\times n}, \omega^{(kn)}), \quad Z \mapsto \left(\sqrt{I^{(k)} - Z\overline{Z}'}, Z\right)$$

is a symplectic embedding with image in $M^0(k,n;\mathbb{C})$, and therefore we get a symplectic embedding $\widehat{\Phi} = \Pr\circ\Phi$ of $(R_I(k,m), \omega^{(km)})$ into $(G(k,n;\mathbb{C}), \sigma^{(k,n)})$.

Proof: Differentiating

$$\Phi(Z)\overline{\Phi(Z)}' = \sqrt{I^{(k)} - Z\overline{Z}'}\sqrt{I^{(k)} - Z\overline{Z}'} + Z\overline{Z}' = I^{(k)}$$

twice on both sides we get

$$d\sqrt{I^{(k)} - Z\overline{Z}'} \bigwedge d\sqrt{I^{(k)} - Z\overline{Z}'} = 0.$$

This leads to

$$d\Phi(Z)\wedge d\overline{\Phi(Z)}' = dZ\wedge d\overline{Z}', \quad \text{i.e.,} \Phi^*\omega^{(kn)} = \omega^{(km)}.$$

Using (45) and (46) we get that the composition $\widehat{\Phi} = \mathrm{Pr}\circ\Phi$ yields the desired symplectic embedding from $(R_I(k,m),\omega^{(km)})$ to $(G(k,n;\mathbb{C}),\sigma^{(k,n)})$. ∎

LEMMA 4.2: *The open unit ball $B^{2km}(1)$ is contained in $R_I(k,m)$.*

Proof: It is well known that for any $Z\in\mathbb{C}^{k\times m}$ with $k\le m$ (resp. $k>m$) there exist unitary matrices U of order k and V of order m such that

$$UZV = (\mathrm{diag}(\lambda_1,\dots,\lambda_k),O) \quad (\text{resp. } UZV = (\mathrm{diag}(\mu_1,\dots,\mu_m),O)')$$

for some $\lambda_1\ge\cdots\ge\lambda_k\ge 0$ (resp. $\mu_1\ge\cdots\ge\mu_m\ge 0$), where $\mathrm{diag}(\lambda_1,\dots,\lambda_k)$ (resp. $\mathrm{diag}(\mu_1,\dots,\mu_m)$) denote the diagonal matrix of order k (resp. m), and O is the zero matrix of order $k\times(m-k)$ (resp. $(k-m)\times m$). Therefore, $Z\in R_I(k,m)$, i.e., $I^{(k)}-Z\overline{Z}'>0$, if and only if $\lambda_j<1$, $j=1,\dots,k$ (resp. $\mu_i<1$, $i=1,\dots,m$). Let $Z\in B^{2km}(1)$. Then

$$\|Z\|^2 = \sum_{i=1}^k\sum_{j=1}^m |z_{ij}|^2 = \mathrm{tr}(Z\overline{Z}') = \sum_{j=1}^m \lambda_j^2 \quad (\text{resp. } \sum_{k=1}^n \mu_k^2) < 1,$$

and thus $\lambda_j<1$ (resp. $\mu_i<1$), i.e., $Z\in R_I(k,m)$. ∎

Now Lemma 4.1 and Lemma 4.2 yield directly

$$(47)\qquad W_G(G(k,n),\sigma^{(k,n)}) \ge W_G(R_I(k,m),\omega^{(km)}) \ge \pi$$

for $m=n-k$. Moreover, for the submanifolds $X^{(k,n)}$ and $Y^{(k,n)}$ of $G(k,n)$ the computation in [SieT, Wi] shows $\Psi_{L^{(k,n)},0,3}(pt;[X^{(k,n)}],[Y^{(k,n)}],pt)=1$. Thus (12) and Theorem 1.13 lead to

$$(48)\quad W_G(G(k,n),\sigma^{(k,n)}) \le C_{HZ}^{(2)}(G(k,n),\sigma^{(k,n)};pt,\alpha) \le \sigma^{(k,n)}(L^{(k,n)}) = \pi$$

for $\alpha=[X^{(k,n)}]$ or $\alpha=[Y^{(k,n)}]$ with $k\le n-2$. Hence the conclusions follow from (47) and (48). Theorem 1.15 is proved. ∎

Proof of Theorem 1.16: Since $\Psi_{L^{(k,n)},0,3}(pt;[X^{(k,n)}],[Y^{(k,n)}],pt)=1$ it follows from Proposition 7.4 that

$$\Psi_{A,0,3}(pt;[M]\times[X^{(k,n)}],[M]\times[Y^{(k,n)}],pt)\ne 0$$

for $A = 0 \times L^{(k,n)}$, where 0 denotes the zero class in $H_2(M;\mathbb{Z})$. Theorem 1.13 implies

$$C_{HZ}^{(2\circ)}(M \times G(k,n), \omega \oplus (a\sigma^{(k,n)}); pt, [M] \times \alpha) \le |a|\pi$$

for $\alpha = [X^{(k,n)}]$ or $\alpha = [Y^{(k,n)}]$ with $k \le n-2$. This implies (20).

For (21) we only prove the case $r = 2$ for the sake of simplicity. The general case is similar. Let us take $A = \bigoplus_{i=1}^{2} L^{(k_i,n_i)} \in H_2(W,\mathbb{Z})$. Then $\Omega(A) = (|a_1| + |a_2|)\pi$. Note that

$$\begin{aligned}\Psi_{L^{(k_i,n_i)},0,3}(pt; pt, [X^{(k_i,n_i)}], [Y^{(k_i,n_i)}]) &= \Psi_{L^{(k_i,n_i)},0,3}(pt; pt, [Y^{(k_i,n_i)}], [X^{(k_i,n_i)}]) \\ &= 1\end{aligned}$$

because the dimensions of $[X^{(k_i,n_i)}]$ and $[Y^{(k_i,n_i)}]$ are even for $i = 1, 2$. Proposition 7.7 gives

$$\begin{aligned}&\Psi_{A,0,3}(pt; pt, [X^{(k_1,n_1)}] \times [Y^{(k_2,n_2)}], [Y^{(k_1,n_1)}] \times [X^{(k_2,n_2)}]) \\ &\quad = \Psi_{L^{(k_1,n_1)},0,3}(pt; pt, [X^{(k_1,n_1)}], [Y^{(k_1,n_1)}], pt) \\ &\qquad \cdot \Psi_{L^{(k_2,n_2)},0,3}(pt; pt, [Y^{(k_2,n_2)}], [X^{(k_2,n_2)}]) = 1, \\ &\Psi_{A,0,3}(pt; pt, [X^{(k_1,n_1)}] \times [X^{(k_2,n_2)}], [Y^{(k_1,n_1)}] \times [Y^{(k_2,n_2)}]) \\ &\quad = \Psi_{L^{(k_1,n_1)},0,3}(pt; pt, [X^{(k_1,n_1)}], [Y^{(k_1,n_1)}]) \\ &\qquad \cdot \Psi_{L^{(k_2,n_2)},0,3}(pt; pt, [X^{(k_2,n_2)}], [Y^{(k_2,n_2)}]) = 1.\end{aligned}$$

As before it follows that

$$\begin{aligned}C_{HZ}^{(2\circ)}(W, \Omega; pt, [X^{(k_1,n_1)}] \times [Y^{(k_2,n_2)}]) &\le \Omega(A) = (|a_1| + |a_2|)\pi, \\ C_{HZ}^{(2\circ)}(W, \Omega; pt, [X^{(k_1,n_1)}] \times [X^{(k_2,n_2)}]) &\le \Omega(A) = (|a_1| + |a_2|)\pi, \\ C_{HZ}^{(2\circ)}(W, \Omega; pt, [Y^{(k_1,n_1)}] \times [Y^{(k_2,n_2)}]) &\le \Omega(A) = (|a_1| + |a_2|)\pi,\end{aligned}$$

proving (21).

To see (22) we assume $r > 1$ because of the result in Theorem 1.15. It immediately follows from (12) and (20) that

$$\mathcal{W}_G(G(k_1,n_1) \times \cdots \times G(k_r,n_r), \sigma^{(k_1,n_1)} \oplus \cdots \oplus \sigma^{(k_r,n_r)}) \le \pi.$$

On the another hand, by Lemma 4.1 we have a symplectic embedding from $(R_I(k_1,n_1) \times \cdots \times R_I(k_r,n_r), \omega^{(k_1n_1)} \oplus \cdots \oplus \omega^{(k_rn_r)})$ to $(G(k_1,n_1) \times \cdots \times G(k_r,n_r), \sigma^{(k_1,n_1)} \oplus \cdots \oplus \sigma^{(k_r,n_r)})$. Moreover, Lemma 4.2 implies that

$$\begin{aligned}B^{2k_1n_1+\cdots+2k_rn_r}(1) &\subset B^{2k_1n_1}(1) \times \cdots \times B^{2k_rn_r}(1) \\ &\subset R_I(k_1,n_1) \times \cdots \times R_I(k_r,n_r).\end{aligned}$$

These give

$$\mathcal{W}_G(G(k_1,n_1)\times\cdots\times G(k_r,n_r),\sigma^{(k_1,n_1)}\oplus\cdots\oplus\sigma^{(k_r,n_r)})\ge\pi$$

and thus desired (22). ∎

Proof of Theorem 1.17: Without loss of generality we may assume $a>0$. Firstly, as in the proof of Theorem 1.16 one shows that

$$\Psi_{A,0,3}(pt;[M\times\mathbb{C}P^n],[M\times pt],pt)\neq 0$$

for $A=[pt\times\mathbb{C}P^1]$, and thus arrive at

$$C^{(2\circ)}_{HZ}(M\times\mathbb{C}P^n,\omega\oplus a\sigma_n;pt,[M\times pt])\le a\pi. \tag{49}$$

Next we prove

$$C^{(2)}_{HZ}(M\times B^{2n}(r),\omega\oplus\omega_0;pt,[M\times pt])=C^{(2)}_{HZ}(M\times B^{2n}(r),\omega\oplus\omega_0;pt,pt). \tag{50}$$

By Definition 1.2 it is clear that the left side in (50) is less than or equal to the right side in (50). To see the converse inequality we take $H\in\mathcal{H}_{ad}(M\times B^{2n}(r),\omega\oplus\omega_0;pt,pt)$. Let $P=P(H)$ and $Q=Q(H)$ be the corresponding submanifolds in Definition 1.2. Since

$$P\subset Q\subset \mathrm{Int}(M\times B^{2n}(r))=M\times\mathrm{Int}(B^{2n}(r))$$

and Q is compact there exists $\eta\in(0,r)$ such that $Q\subset M\times B^{2n}(\eta)$. (Note that here we use $\partial M=\emptyset$.) Therefore, H may be viewed as an element of $\mathcal{H}_{ad}(M\times B^{2n}(r),\omega\oplus\omega_0;pt,[M\times pt])$ naturally. This implies that the left side in (50) is more than or equal to the right side in (50).

Thirdly, as in [HZ1, HZ2] one proves

$$C^{(2)}_{HZ}(M\times B^{2n}(r),\omega\oplus\omega_0;pt,pt)\ge\pi r^2 \tag{51}$$

for any $r>0$. By (49), Theorem 1.5 (v), (50) and (51) we can obtain

$$\begin{aligned}a\pi &\ge C^{(2\circ)}_{HZ}(M\times\mathbb{C}P^n,\omega\oplus a\sigma_n;pt,[M\times pt])\\ &\ge C^{(2)}_{HZ}(M\times\mathbb{C}P^n,\omega\oplus a\sigma_n;pt,[M\times pt])\\ &\ge C^{(2)}_{HZ}(M\times B^{2n}(\delta\sqrt{a}),\omega\oplus\omega_0;pt,[M\times pt])\\ &= C^{(2)}_{HZ}(M\times B^{2n}(\delta\sqrt{a}),\omega\oplus\omega_0;pt,pt)\\ &\ge\pi\delta^2 a\end{aligned}$$

for any $\delta \in (0,1)$. Here we use the symplectic embedding $(B^{2n}(\delta\sqrt{a}), \omega_0) \hookrightarrow (\mathbb{C}P^n, a\sigma_n)$ in the proof of Corollary 1.5 in [HV2] for any $0 < \delta < 1$. Taking $\delta \to 1$, we find that for $\delta = 1$ the above inequalities are equalities. Together with Lemma 1.4 we obtain (23) and $C(M \times B^{2n}(r), \omega \oplus \omega_0) = \pi r^2$ in (24).

To prove the other equality of (24), i.e., $C(M \times Z^{2n}(r), \omega \oplus \omega_0) = \pi r^2$, note that each $H \in \mathcal{H}_{ad}(M \times Z^{2n}(r), \omega \oplus \omega_0; pt, pt)$ can naturally be viewed as a function in $\mathcal{H}_{ad}(M \times B^2(r) \times \mathbb{R}^{2n-2}/m\mathbb{Z}^{2n-2}, \omega \oplus \omega_0 \oplus \omega_{st}; pt, pt)$ for sufficiently large $m > 0$. Here ω_{st} is the standard symplectic structure on the tours $\mathbb{R}^{2n-2}/m\mathbb{Z}^{2n-2}$. It follows from the equality just proved in (24) that $\max H \leq \pi r^2$ and so

$$C_{HZ}^{(2\circ)}(M \times Z^{2n}(r), \omega \oplus \omega_0; pt, pt) \leq \pi r^2$$

for any $r > 0$. The desired conclusions easily follow. ∎

In order to prove Theorem 1.21 we need the following lemma told to me by Professor Dusa McDuff and Dr. Felix Schlenk.

LEMMA 4.3: *For any two closed symplectic manifolds (M, ω) and (N, σ),*

$$c(M \times N, \omega \oplus \sigma) \geq c(M, \omega) + c(N, \sigma)$$

for $c = c_{HZ}$, c_{HZ}° and C_{HZ}, C_{HZ}°.

According to Lemma 1.4 it suffices to prove Lemma 4.3 for c_{HZ} and c_{HZ}°. Let F and G be admissible functions on M and N, respectively. Since the Hamiltonian system for $F+G$ splits, we see that $F+G$ is an admissible function on $M \times N$. From this Lemma 4.3 follows at once.

Proof of Theorem 1.21: We denote by (W, ω) the product manifold in Theorem 1.21. Without loss of generality we may assume $a_i > 0$, $i = 1, \dots, k$. Let $A_i = [\mathbb{C}P^1]$ be the generators of $H_2(\mathbb{C}P^{n_i}; \mathbb{Z})$, $i = 1, \dots, k$. They are indecomposable classes. Since $[Y^{(1,n_i)}] = pt$, it follows from the proof of Theorem 1.16 that

$$\Psi_{A_i,0,3}(pt; pt, pt, [X^{(1,n_i)}]) = 1$$

for $i = 1, \dots, k$. Set $A = A_1 \times \cdots \times A_k$. Note that each $(\mathbb{C}P^{n_i}, a_i\sigma_{n_i})$ is monotone. By Proposition 7.7 in the appendix we have

$$\Psi_{A,0,3}(pt; pt, pt, \beta) = 1$$

for some class $\beta \in H_*(W, \mathbb{Q})$. Thus by Corollary 1.19 we get that

$$c(W, \omega) \leq \omega(A) = (a_1 + \cdots + a_k)\pi \tag{52}$$

for $c = c_{HZ}$, c°_{HZ}. On the other hand, Lemma 4.3 yields

$$c(W,\omega) \geq \sum_{i=1}^{k} c(\mathbb{C}P^{n_i}, a_i\sigma_{n_i}) = (a_1 + \cdots + a_k)\pi$$

for $c = c_{HZ}$, c°_{HZ}. ∎

5. Proof of Theorems 1.22 and 1.24

Proof of Theorem 1.22: Under the assumptions of Theorem 1.22 it follows from Remark 1.11 that the Gromov–Witten invariant

$$\Psi_{A,g,m+2}(\pi^*C; \alpha_0, PD([\omega]), \alpha_1, \ldots, \alpha_m) \neq 0,$$

and thus Theorem 1.10 leads to

$$C^{(2)}_{HZ}(M, \omega; \alpha_0, PD([\omega])) < +\infty.$$

For a sufficiently small $\epsilon > 0$ the well-known Lagrangian neighborhood theorem due to Weinstein [We1] yields a symplectomorphism ϕ from $(U_\epsilon, \omega_{\text{can}})$ to a neighborhood of L in (M, ω) such that $\phi|_L = id$. Since L is a Lagrange submanifold one can, as in [Lu3, V6], use the Poincaré–Lefschetz duality theorem to prove that there exists a cycle representative of $PD([\omega])$ whose support is contained in $M \setminus \phi(U_\epsilon)$ because ω is exact near L. By (6) we get that

$$\begin{aligned} C^{(2)}_{HZ}(U_\epsilon, \omega_{\text{can}}; \tilde{\alpha}_0, pt) &= C^{(2)}_{HZ}(\phi(U_\epsilon), \omega; \tilde{\alpha}_0, pt) \\ &\leq C^{(2)}_{HZ}(M, \omega; \alpha_0, PD([\omega])) < +\infty. \end{aligned} \tag{53}$$

Here we still denote by $\tilde{\alpha}_0$ the images in $H_*(U_\varepsilon, \mathbb{Q})$ and $H_*(\phi(U_\varepsilon), \mathbb{Q})$ of $\tilde{\alpha}_0$ under the maps induced by the inclusions $L \hookrightarrow U_\varepsilon$ and $L \hookrightarrow \phi(U_\varepsilon)$. Note that for any $\lambda \neq 0$ the map

$$\Phi_\lambda \colon T^*L \to T^*L, \quad (q, v^*) \mapsto (q, \lambda v^*),$$

satisfies $\Phi_\lambda^*\omega_{\text{can}} = \lambda\omega_{\text{can}}$. Theorem 1.5 (iv), (53) and this fact imply that

$$C^{(2)}_{HZ}(U_c, \omega_{\text{can}}; \tilde{\alpha}_0, pt) < +\infty$$

for any $c > 0$.

In the case $g = 0$, since the inclusion $L \hookrightarrow M$ induces an injective homomorphism $\pi_1(L) \to \pi_1(M)$ and thus $\phi(U_\epsilon) \hookrightarrow M$ also induces an injective homomorphism $\pi_1(\phi(U_\varepsilon)) \to \pi_1(M)$ it follows from (8) that

$$C_{HZ}^{(2\circ)}(U_\epsilon, \omega_{\mathrm{can}}; \tilde{\alpha}_0, pt) = C_{HZ}^{(2\circ)}(\phi(U_\epsilon), \omega; \tilde{\alpha}_0, pt) \\ \leq C_{HZ}^{(2\circ)}(M, \omega; \alpha_0, PD([\omega])) < +\infty,$$

and thus that $C_{HZ}^{(2\circ)}(U_c, \omega_{\mathrm{can}}; \tilde{\alpha}_0, pt) < +\infty$ for any $c > 0$.

In particular, if L is a Lagrange submanifold of a g-symplectic uniruled manifold (M, ω), then we can take $\alpha_0 = pt$ and derive from (7)

$$c_{HZ}(U_c, \omega_{\mathrm{can}}) = C_{HZ}(U_c, \omega_{\mathrm{can}}) < +\infty$$

for any $c > 0$, and from (9)

$$c_{HZ}^{\circ}(U_c, \omega_{\mathrm{can}}) = C_{HZ}^{\circ}(U_c, \omega_{\mathrm{can}}) < +\infty$$

for all $c > 0$ if $g = 0$ and the inclusion $L \hookrightarrow M$ induces an injective homomorphism $\pi_1(L) \to \pi_1(M)$. Here we use Lemma 1.4 and the fact that U_c is a compact smooth manifold with connected boundary and of codimension zero because $\dim L \geq 2$.

To see the final claim note that $(M, -\omega)$ is also strong g-symplectic uniruled. It follows from Proposition 7.5 that the product $(M \times M, (-\omega) \oplus \omega)$ is strong 0-symplectic uniruled. By the Lagrangian neighborhood theorem there exists a neighborhood $\mathcal{N}(\triangle) \subset M \times M$ of the diagonal $\triangle$, a fiberwise convex neighborhood $\mathcal{N}(M_0) \subset T^*M$ of the zero section M_0, and a symplectomorphism ψ: $(\mathcal{N}(\triangle), (-\omega) \oplus \omega) \to (T^*M, \omega_{\mathrm{can}})$ such that $\psi(x, x) = (x, 0)$ for $x \in M$. Note also that the inclusion $\triangle \hookrightarrow M \times M$ induces an injective homomorphism $\pi_1(\triangle) \to \pi_1(M \times M)$. The desired conclusion follows immediately. ∎

Proof of Theorem 1.24: The case $\dim M = 2$ is obvious. So we assume that $\dim M \geq 4$. We follow [Bi1]. Let p: $L = \nu(N) \to N$ be the symplectic normal bundle of N in (M, ω). It may naturally be viewed as a complex line bundle with an obvious S^1-action

$$t \cdot (b, v) = (b, e^{2\pi i t} v), \quad (b, v) \in L \quad \text{and} \quad t \in S^1 = \mathbb{R}/\mathbb{Z}.$$

Consider the projectivized bundle π: $\mathbf{P}(L \oplus \mathbb{C}) \to N$ whose fiber at $b \in N$ is the complex projective space $\mathbf{P}(L_b \oplus \mathbb{C})$. This bundle has a natural S^1-action induced by the action $t \cdot z = e^{-2\pi i t} z$ of S^1 on each fiber summand of $\mathbb{C}$, i.e.,

$t \cdot (b, [v : z]) = (b, [v : e^{-2\pi i t} z])$. It has also two special sections, the zero section $Z_0 = \mathbf{P}(\{0\} \oplus \mathbb{C})$ and the infinity section $Z_\infty = \mathbf{P}(L \oplus \{0\})$. One can construct an S^1-invariant symplectic form on $\mathbf{P}(L \oplus \mathbb{C})$. Roughly speaking, fix any Hermitian metric $\|\cdot\|$ on L and denote by $p_N : S(L) = \{(b, v) \in L | \|v\| = 1\} \to N$ the associated unit circle bundle of L. The latter is a principal S^1-bundle. Let $S^1 = \mathbb{R}/\mathbb{Z}$ act on $\mathbb{C}P^1$ and $S(L) \times \mathbb{C}P^1$ by

$$t \cdot [z_0 : z_1] = [z_0 : e^{-2\pi i t} z_1], \quad [z_0 : z_1] \in \mathbb{C}P^1,$$
$$t \cdot ((b, v), [z_0 : z_1]) = ((b, e^{-2\pi i t} v), [z_0 : e^{-2\pi i t} z_1])$$

for $(b, v) \in S(L)$ and $t \in S^1$. Then the quotient manifold $S(L) \times_{S^1} \mathbb{C}P^1$ and $\mathbf{P}(L \oplus \mathbb{C})$ can be identified via the diffeomorphism induced by the projection

$$\widetilde{\Phi}\colon S(L) \times \mathbb{C}P^1 \to \mathbf{P}(L \oplus \mathbb{C}), \quad ((b, v), [z_0 : z_1]) \mapsto (b, [z_0 v : z_1]).$$

Under this identification one has $Z_0 = S(L) \times_{S^1} \{[0 : 1]\}$ and $Z_\infty = S(L) \times_{S^1} \{[1 : 0]\}$. If R^∇ is the curvature of the Hermitian connection ∇ on L, then $\rho_N := \frac{1}{2\pi i} R^\nabla$ is a representing 2-form of the Chern class $c_1(L)$. Choose $0 < \lambda_0 < \varepsilon$ so that

$$\tau_\lambda := \omega|_N + \lambda \rho_N$$

are symplectic forms on N for all $0 < \lambda \le \lambda_0$. Let $h\colon \mathbb{C}P^1 \to [0, 1]$ be given by $h([z_0 : z_1]) = |z_0|/(|z_0|^2 + |z_1|^2)$. Define a map

$$H_{\lambda_0}\colon S(L) \times_{S^1} \mathbb{C}P^1 \to [0, \lambda_0], [(b, v), [z_0 : z_1]] \mapsto \lambda_0 h([z_0 : z_1]).$$

Then all level sets $H_{\lambda_0}^{-1}(\lambda)$, $\lambda \notin \{0, \lambda_0\}$, are diffeomorphic to $S(L)$, and the only critical submanifolds of H_{λ_0} are $H_{\lambda_0}^{-1}(0) = Z_0$ and $H_{\lambda_0}^{-1}(\lambda_0) = Z_\infty$. As in Example 5.10 in [McSa1] (see also [MWo]) one gets an S^1-invariant symplectic form ω_{λ_0} on $\mathbf{P}(L \oplus \mathbb{C})$ such that Z_0, Z_∞ and all fibers are symplectic submanifolds. More precisely, $\omega_{\lambda_0}|_{Z_0} = \omega|_N$, $\omega_{\lambda_0}|_{Z_\infty} = \omega|_N + \lambda_0 \rho_N$ and each fiber $\mathbf{P}(L \oplus \mathbb{C})_b \equiv \mathbb{C}P^1$ is equipped with an S^1-invariant symplectic form with corresponding moment map $\lambda_0 h$, i.e., $\lambda_0 \omega_{\mathrm{FS}}$. Here ω_{FS} is the standard Fubini–Study form on $\mathbb{C}P^1$ with $\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = 1$. Furthermore, Z_0 has normal bundle in $(\mathbf{P}(L \oplus \mathbb{C}), \omega_{\lambda_0})$ with first Chern class $[\rho_N] = c_1(L)$, see the appendix in [MWo].

As in [Bi1], from a transgression 1-form α^∇ of the connection ∇ on $L \setminus 0$ one can get a 1-form $\alpha \in \Omega^1(S(L))$ such that $d\alpha = -p_N^* \rho_N$ and $\alpha(X) \equiv 1$, where $X\colon S(L) \to TS(L)$ is the fundamental vector field of the above S^1-action on $S(L)$. The first condition means that $\mathcal{H} = \mathrm{Ker}(\alpha)$ is the horizontal distribution of the connection on $S(L)$ induced by ∇. By Exercise 5.11 in [McSa1], under the above identification $\mathbf{P}(L \oplus \mathbb{C}) = S(L) \times_{S^1} \mathbb{C}P^1$, the symplectic form ω_{λ_0} is induced by the S^1-invariant closed 2-form

$$\Omega_{\lambda_0} := p_N^*(\omega|_N) - \lambda_0 d(h\alpha) + \lambda_0 \omega_{\rm FS}$$

on $S(L) \times \mathbb{C}P^1$.

Set $X_L = \mathbf{P}(L \oplus \mathbb{C})$. Take an almost complex structure $J_N \in \mathcal{J}(N, \omega|_N)$. By shrinking $\lambda_0 > 0$ we can assume that J_N is $(\omega|_N + \lambda\rho_N)$-tame for all $0 < \lambda \leq \lambda_0$, i.e., $J_N \in \mathcal{J}_\tau(N, \omega|_N + \lambda\rho_N)$. (This is not needed in the case of [Bi1] because ρ_N can be taken as $\omega|_N$ there.) Notice that the above horizontal distribution $\mathcal{H}$ over $S(L)$ naturally induces a horizontal distribution $\widetilde{\mathcal{H}} = \widetilde{\Phi}_*(\mathcal{H} \times 0)$ on X_L. So $TX_L = \widetilde{\mathcal{H}} \oplus \widetilde{\mathcal{V}}$, where $\widetilde{\mathcal{V}} \subset TX_L$ is the vertical subbundle whose fiber at $q \in X_L$ is $\widetilde{\mathcal{V}}_q = {\rm Ker}(d\pi(q)) = T_q(X_L)_{\pi(q)}$. Actually $\widetilde{\mathcal{H}}$ is exactly the horizontal distribution on $\mathbf{P}(L \oplus \mathbb{C})$ induced by the sum of the connection ∇ on L and the trivial connection on $\mathbb{C} \to N$. Since $\Omega_{\lambda_0} = p_N^*(\omega|_N + \lambda_0 h\rho_N) + \lambda_0(\omega_{\rm FS} - dh \wedge \alpha)$, it is not hard to check that for any $q \in X_L$ the subspaces $\widetilde{\mathcal{H}}_q$ and $\widetilde{\mathcal{V}}_q$ are $(\omega_{\lambda_0})_q$-orthogonal. Similar to [Bi1] we construct an almost complex structure J_X on X_L as follows; for any $q \in X_L$, $J_X|_{\widetilde{\mathcal{H}}_q}$ is the horizontal lift of $(J_N)_q$ by the linear isomorphism $d\pi(q)|_{\widetilde{\mathcal{H}}_q}\colon \widetilde{\mathcal{H}}_q \to T_{\pi(q)}N$, and the restriction of J_X to the fiber $(X_L)_{\pi(q)} = \mathbf{P}(L_{\pi(q)} \oplus \mathbb{C})$ is the sum of the complex structure determined by the Hermitian metric $\|\cdot\|$ on L and of the standard one on $\mathbb{C}$. This J_X is ω_{λ_0}-tame because $J_N \in \mathcal{J}_\tau(N, \omega|_N + \lambda\rho_N)$ for all $0 < \lambda \leq \lambda_0$. One easily sees that the almost complex structure J_X is **fibred** on X_L in the sense of Definition 2.2 of [Mc2]. Hence with J_X we can prove as in Lemma 2.3 of [Lu6] that for the homology class $F \in H_2(X_L; \mathbb{Z})$ of a fibre of $X_L \to N$ the Gromov-Witten invariant

$$\Psi_{F,0,3}^{(X_L, \omega_{\lambda_0})}(pt; [Z_0], [Z_\infty], pt) = 1.$$

That is, $(X_L, \omega_{\lambda_0})$ is a strong 0-symplectic uniruled manifold in the sense of Definition 1.14. By Theorem 1.10 we have

$$C_{HZ}^{(2\circ)}(X_L, \omega_{\lambda_0}; pt, [Z_\infty]) \leq {\rm GW}_0(X_L, \omega_{\lambda_0}; pt, [Z_\infty]) \leq \omega_{\lambda_0}(F) = \lambda_0.$$

Note that for any $0 < \delta < \lambda_0$ the set $\{H_{\lambda_0} \leq \delta\}$ is a smooth compact submanifold in X_L with connected boundary and of codimension zero that is a neighborhood of Z_0 in X_L. It is easily seen that the inclusion $\{H_{\lambda_0} \leq \delta\} \subset X_L$ induces an injective homomorphism $\pi_1(\{H_{\lambda_0} \leq \delta\}) \hookrightarrow \pi_1(X_L)$. It follows from (9) that

$$c_{HZ}^\circ(\{H_{\lambda_0} \leq \delta\}, \omega_{\lambda_0}) \leq C_{HZ}^{(2\circ)}(X_L, \omega_{\lambda_0}; pt, [Z_\infty]) \leq \lambda_0.$$

Identifying N with the zero section 0_L and thus Z_0 it follows from the symplectic neighborhood theorem that for $\delta > 0$ sufficiently small, $(\{H_{\lambda_0} \le \delta\}, \omega_{\lambda_0})$ is symplectomorphic to a smooth compact submanifold $W \subset M$ with connected boundary and of codimension zero that is a neighborhood of N in M. Together with Lemma 1.4 we therefore get

$$c^{\circ}_{HZ}(W,\omega) = C^{\circ}_{HZ}(W,\omega) \le \lambda_0 < \varepsilon.$$

The desired conclusion is proved.

6. Proof of Theorem 1.35

The idea is the same as in [Ka]. We can assume that $n/k \ge 2$. Following the notations in the proof of Theorem 1.15, notice that the canonical atlas on $G(k,n)$ given by (44) has $\binom{n}{k}$ charts, and that for each chart

$$(\Theta(\alpha_1,\dots,\alpha_k), U(\alpha_1,\dots,\alpha_k))$$

Lemma 4.1 yields a symplectic embedding $\widehat{\Phi}_{\alpha_1\cdots\alpha_k}$ of $(R_I(k,m),\omega^{(km)})$ into $(G(k,n),\sigma^{(k,n)})$ given by

$$Z \mapsto [(\sqrt{I^{(k)} - Z\overline{Z}'}, Z)P(\alpha_1,\dots,\alpha_k)],$$

where $P(\alpha_1,\dots,\alpha_k)$ is the $n\times n$ permutation matrix such that (43) holds for the matrix $B = (I^{(k)}, Z)P(\alpha_1,\dots,\alpha_k)$. Moreover, for the matrix $A = (\sqrt{I^{(k)} - Z\overline{Z}'}, Z)P(\alpha_1,\dots,\alpha_k)$ we have

$$A_{\alpha_1\cdots\alpha_k} = \sqrt{I^{(k)} - Z\overline{Z}'} \quad \text{and} \quad A_{\alpha_{k+1}\cdots\alpha_n} = Z.$$

Note that

$$\begin{aligned}\|A\|^2 &= \|A_{\alpha_1\cdots\alpha_k}\|^2 + \|A_{\alpha_{k+1}\cdots\alpha_n}\|^2\\ &= \mathrm{tr}(A_{\alpha_1\cdots\alpha_k}\overline{A}'_{\alpha_1\cdots\alpha_k}) + \mathrm{tr}(A_{\alpha_{k+1}\cdots\alpha_n}\overline{A}'_{\alpha_{k+1}\cdots\alpha_n})\\ &= \mathrm{tr}(I^{(k)} - Z\overline{Z}') + \mathrm{tr}(Z\overline{Z}') = k,\end{aligned}$$

and therefore

$$\|A_{\alpha_1\cdots\alpha_k}\|^2 = k - \|A_{\alpha_{k+1}\cdots\alpha_n}\|^2 = k - \|Z\|^2.$$

By Lemma 4.2 these show that $\widehat{\Phi}_{\alpha_1\cdots\alpha_k}(B^{2km}(r))$ is contained in

$$\begin{aligned}&\Lambda(\alpha_1,\dots,\alpha_k;r)\\ &= \{[B]\in G(k,n) \mid \text{ for all } A\in[B]\cap M^0(k,n;\mathbb{C}), \|A_{\alpha_1\cdots\alpha_k}\|^2 > k - r^2\}\end{aligned} \tag{54}$$

for any $0 < r \le 1$. Note that $k > 1$ and $n/k \ge 2$. There must be two disjoint subsets of $\{1, \dots, n\}$, say $\{\alpha_1, \dots, \alpha_k\}$ and $\{\beta_1, \dots, \beta_k\}$, such that $\alpha_1 < \cdots < \alpha_k$ and $\beta_1 < \cdots < \beta_k$. For any two such subsets we claim that

$$\widehat{\Phi}_{\alpha_1\cdots\alpha_k}(B^{2km}(1)) \cap \widehat{\Phi}_{\beta_1\cdots\beta_k}(B^{2km}(1)) = \emptyset.$$

In fact, $\Lambda(\alpha_1, \dots, \alpha_k; 1)$ and $\Lambda(\beta_1, \dots, \beta_k; 1)$ are disjoint. Otherwise, let $[B]$ belong to their intersection and take a representative A of $[B]$ in $M^0(k, n; \mathbb{C})$. Then

$$k \ge \|A_{\alpha_1\cdots\alpha_k}\|^2 + \|A_{\beta_1\cdots\beta_k}\|^2 > 2k - 2$$

by (54). This contradicts the assumption that $k \ge 2$. Now the conclusion follows from the fact that there exist exactly $[n/k]$ mutually disjoint subsets of $\{1, \dots, n\}$ consisting of k numbers. ∎

Proof of (27): Notice that $G(k, n)$ can be embedded into the complex projective space $\mathbb{C}P^N$ with $N = n!/(n-k)!k! - 1$ by the Plücker map p ([GH]), and that for any l-dimensional subvariety X of $\mathbb{C}P^N$ one has

$$\mathrm{Vol}(X) = \deg(X) \cdot \mathrm{Vol}(L)$$

with respect to the Fubini–Study metric, where L is an l-dimensional linear subspace of $\mathbb{C}P^N$ (cf. [Fu, p. 384]). But it was shown in Example 14.7.11 of [Fu] that

$$\deg(p(G(k, n))) = \frac{1! \cdot 2! \cdots (k-1)! \cdot (k(n-k))!}{(n-k)! \cdot (n-k+1)! \cdots (n-1)!}.$$

It is well-known that the volume of a $k(n-k)$-dimensional linear subspace L of $\mathbb{C}P^N$ is

$$\mathrm{Vol}(\mathbb{C}P^{k(n-k)}) = \frac{\pi^{k(n-k)}}{(k(n-k))!}.$$

These give (27). ∎

7. Appendix: The Gromov–Witten invariants of product manifolds

In this appendix we collect some results on Gromov–Witten invariants needed in this paper. They either are easily proved or follow from the references given.

Let (V, ω) be a closed symplectic manifold of dimension $2n$. Recall that for a given class $A \in H_2(V; \mathbb{Z})$ the Gromov–Witten invariant of genus g and with k marked points is a homomorphism

$$\Psi^V_{A,g,k}\colon H_*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q}) \times H_*(V; \mathbb{Q})^k \to \mathbb{Q},$$

where $2g+k \geq 3$ and $\overline{\mathcal{M}}_{g,k}$ is the space of isomorphism classes of genus g stable curves with k marked points, which is a connected Kähler orbifold of complex dimension $3g-3+k$. In [Lu8] we used the cohomology $H^*_c(V;\mathbb{Q})$ with compact support and the different notation $\mathcal{GW}^{(\omega,\mu,J)}_{A,g,k}$ to denote the GW-invariants, since we also considered noncompact symplectic manifolds for which the dependence on further data needs to be indicated. For closed symplectic manifolds we easily translate the composition law and reduction formulas in [Lu8] into the homology version, which is the same as the ones in [RT2]. Let integers $g_i \geq 0$ and $k_i > 0$ satisfy $2g_i + k_i \geq 3$, $i=1,2$. Set $g = g_1 + g_2$ and $k = k_1 + k_2$ and fix a decomposition $S = S_1 \cup S_2$ of $\{1,\ldots,k\}$ with $|S_i| = k_i$. Then there is a canonical embedding

$$\theta_S\colon \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \to \overline{\mathcal{M}}_{g,k}, \tag{55}$$

which assigns to marked curves $(\Sigma_i; x^i_1,\ldots,x^i_{k_i+1})$, $i=1,2$, their union $\Sigma_1 \cup \Sigma_2$ with $x^1_{k_1+1}$ and $x^2_{k_2+1}$ identified and the remaining points renumbered by $\{1,\ldots,k\}$ according to S. Let

$$\mu_{g,k}\colon \overline{\mathcal{M}}_{g-1,k+2} \to \overline{\mathcal{M}}_{g,k}$$

be the map corresponding to gluing together the last two marked points. It is continuous. Suppose that $\{\beta_b\}_{b=1}^L$ is a homogeneous basis of $H_*(V;\mathbb{Z})$ modulo torsion, (η_{ab}) its intersection matrix and $(\eta^{ab}) = (\eta_{ab})^{-1}$.

Composition law: *Let*

$$[K_i] \in H_*(\overline{\mathcal{M}}_{g_i,k_i+1};\mathbb{Q}), \quad i=1,2, \quad [K_0] \in H_*(\overline{\mathcal{M}}_{g-1,k+2};\mathbb{Q})$$

and $A \in H_2(V;\mathbb{Z})$. Then for any $\alpha_1,\ldots,\alpha_k$ in $H_*(V;\mathbb{Q})$ we have

$$\Psi^V_{A,g,k}(\theta_{S*}([K_1 \times K_2]);\alpha_1,\ldots,\alpha_k) = (-1)^{\mathrm{cod}(K_2)\sum_{i=1}^{k_1}\mathrm{cod}(\alpha_i)}$$

$$\sum_{A=A_1+A_2}\sum_{a,b}\Psi^V_{A_1,g_1,k_1+1}([K_1];\{\alpha_i\}_{i\leq k_1},\beta_a)\eta^{ab}\Psi^V_{A_2,g_2,k_2+1}([K_2];\beta_b,\{\alpha_j\}_{j>k_1}),$$

$$\Psi^V_{A,g,k}((\mu_{g,k})_*([K_0]);\alpha_1,\ldots,\alpha_k) = \sum_{a,b}\Psi^V_{A,g-1,k+2}([K_0];\alpha_1,\ldots,\alpha_k,\beta_a,\beta_b)\eta^{ab}.$$

Remark that $(-1)^{\mathrm{cod}(K_2)\sum_{i=1}^{k_1}\mathrm{cod}(\alpha_i)} = (-1)^{\dim(K_2)\sum_{i=1}^{k_1}\dim(\alpha_i)}$ because the dimensions of $\overline{\mathcal{M}}_{g_i,k_i+1}$ and V are even. Denote the map forgetting the last marked point by

$$\pi_k\colon \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k-1}.$$

REDUCTION FORMULA: *Suppose that $(g,k) \neq (0,3),(1,1)$. Then*

(i) *for any $\alpha_1,\ldots,\alpha_{k-1}$ in $H_*(V;\mathbb{Q})$ and $[K] \in H_*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})$ we have*

(56)
$$\Psi^V_{A,g,k}([K];\alpha_1,\ldots,\alpha_{k-1},[V]) = \Psi^V_{A,g,k-1}((\pi_k)_*([K]);\alpha_1,\ldots,\alpha_{k-1});$$

(ii) *if $\alpha_k \in H_{2n-2}(V;\mathbb{Q})$ we have*

(57)
$$\Psi^V_{A,g,k}([\pi_k^{-1}(K)];\alpha_1,\ldots,\alpha_k) = PD(\alpha_k)(A)\Psi^V_{A,g,k-1}([K];\alpha_1,\ldots,\alpha_{k-1}).$$

LEMMA 7.1: *Let (V,ω) be a closed symplectic manifold, $\{\beta_b\}_{b=1}^L$ a homogeneous basis of $H_*(V;\mathbb{Z})$ modulo torsion as in the composition law above. Suppose that there exist homology classes $A \in H_2(V;\mathbb{Z})$, $\alpha_1,\ldots,\alpha_m \in H_*(V;\mathbb{Q})$ and $g > 0$ such that*

$$\Psi^V_{A,g,m}(pt;\alpha_1,\ldots,\alpha_m) \neq 0. \tag{58}$$

Then for each nonnegative integer $g' < g$ we have

$$\Psi^V_{A,g',m+2s}(pt;\alpha_1,\ldots,\alpha_m,\beta_{a_1},\beta_{b_1},\cdots,\beta_{a_s},\beta_{b_s}) \neq 0$$

for $s = g - g'$ and some β_{a_i},β_{b_i} in $\{\beta_b\}_{b=1}^L$, $i = 1,\ldots,s$.

Proof: By the composition law for Gromov–Witten invariants we have

$$\begin{aligned}\Psi^V_{A,g,m}(pt;\alpha_1,\ldots,\alpha_m) &= \Psi^V_{A,g,m}((\mu_{g,m})_*(pt);\alpha_1,\ldots,\alpha_m)\\ &= \sum_{a,b}\Psi^V_{A,g-1,m+2}(pt;\alpha_1,\ldots,\alpha_m,\beta_a,\beta_b)\eta^{ab}.\end{aligned}$$

By (58), the left side is not equal to zero. So there exists a pair (a,b) such that

$$\Psi^V_{A,g-1,m+2}(pt;\alpha_1,\ldots,\alpha_m,\beta_a,\beta_b) \neq 0.$$

If $g-1 > g'$ we can repeat this argument to reduce $g-1$. After $s = g-g'$ steps the lemma follows. ∎

LEMMA 7.2: *Let (V,ω) and $\{\beta_b\}_{b=1}^L$ be as in Lemma 7.1. Suppose that there exist homology classes*

$$A \in H_2(V;\mathbb{Z}),\xi_1,\ldots,\xi_k \in H_*(V;\mathbb{Q}) \quad \text{and} \quad [K] \in H_*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})$$

such that

$$\Psi^V_{A,g,k}([K];\xi_1,\ldots,\xi_k) \neq 0 \tag{59}$$

for some integer $g \geq 0$. Then for each integer $m > k$ we have

$$\Psi^V_{A,g,m}([K_1];\xi_1,\ldots,\xi_k,\overbrace{PD([\omega]),\ldots,PD([\omega])}^{m-k}) \neq 0.$$

Here $K_1 = (\pi_m)^{-1} \circ \cdots \circ (\pi_{k+1})^{-1}(K)$.

Proof: Using the definition of the GW-invariants, it follows from (59) that $2g + k \geq 3$ and that the space $\overline{\mathcal{M}}_{g,k}(V, J, A)$ of k-pointed stable J-maps of genus g and of class A in V is nonempty for generic $J \in \mathcal{J}(V,\omega)$. In particular, this implies $\omega(A) \neq 0$. Applying the reduction formula (57) to (59) we have

$$\Psi^V_{A,g,k+1}([\pi^{-1}_{k+1}(K)];\xi_1,\ldots,\xi_k,PD([\omega])) = \omega(A)\cdot\Psi^V_{A,g,k}([K];\xi_1,\ldots,\xi_k) \neq 0.$$

Continuing this process $m - k - 1$ times again we get the desired conclusion. ∎

PROPOSITION 7.3: *For a closed symplectic manifold (V,ω), if there exist homology classes $A \in H_2(V;\mathbb{Z})$ and $\alpha_i \in H_*(V;\mathbb{Q})$, $i = 1,\ldots,k$, such that the Gromov–Witten invariant*

$$\Psi_{A,g,k+1}(pt;pt,\alpha_1,\ldots,\alpha_k) \neq 0 \tag{60}$$

for some integer $g \geq 0$, then there exist homology classes $B \in H_2(V;\mathbb{Z})$ and $\beta_1,\beta_2 \in H_*(V;\mathbb{Q})$ such that

$$\Psi_{B,0,3}(pt;pt,\beta_1,\beta_2) \neq 0. \tag{61}$$

Consequently, every strong symplectic uniruled manifold is strong 0-symplectic uniruled.

(61) implies that B is spherical. In fact, in this case there exists a 3-pointed stable J-curve of genus zero and in class B. By the gluing arguments we can get a J-holomorphic sphere $f: \mathbb{C}P^1 \to M$ which represents the class B. That is, B is J-effective. So B is necessarily spherical; cf. page 67 in [McSa2].

Proof of Proposition 7.3: By Lemma 7.1, we can assume $g = 0$ in (60), i.e.,

$$\Psi_{A,0,k+1}(pt;pt,\alpha_1,\ldots,\alpha_k) \neq 0. \tag{62}$$

This implies that $k + 1 \geq 3$ or $k \geq 2$. If $k = 2$ then the conclusion holds. If $k = 3$ we can use the reduction formula (56) to get

$$\Psi_{A,0,5}(pt;pt,\alpha_1,\ldots,\alpha_3,[V]) = \Psi_{A,0,4}(pt;pt,\alpha_1,\ldots,\alpha_3) \neq 0.$$

Therefore we can actually assume that $k \geq 4$ in (62). Since $\overline{\mathcal{M}}_{0,m}$ is connected for every integer $m \geq 3$, $H_0(\overline{\mathcal{M}}_{0,m}, \mathbb{Q})$ is generated by pt. For the canonical embedding θ_S as in (55) we have $\theta_{S*}(pt \times pt) = pt$. Hence it follows from the composition law that

$$\Psi_{A,0,k+1}(pt; pt, \alpha_1, \ldots, \alpha_k)$$
$$= \sum_{A=A_1+A_2} \sum_{a,b} \Psi_{A_1,0,4}(pt; pt, \alpha_1, \alpha_2, \beta_a)\eta^{ab}\Psi_{A_2,0,k-1}(pt; \beta_b, \alpha_3, \ldots, \alpha_k)$$

because $\mathrm{cod}(K_2) = \mathrm{cod}(pt)$ is even. This implies that

$$\Psi_{A_1,0,4}(pt; pt, \alpha_1, \alpha_2, \beta_a) \neq 0 \tag{63}$$

for some $A_1 \in H_2(V; \mathbb{Z})$ and $1 \leq a \leq L$. By the associativity of the quantum multiplication,

$$\Psi_{A_1,0,4}(pt; pt, \alpha_1, \alpha_2, \beta_a) =$$
$$\pm \sum_{A_1=A_{11}+A_{12}} \sum_{l} \Psi_{A_{11},0,3}(pt; pt, \alpha_1, e_l)\Psi_{A_{12},0,3}(pt; f_l, \alpha_2, \beta_a)$$

where $\{e_l\}_l$ is a basis for the homology $H_*(M; \mathbb{Q})$ and $\{f_l\}_l$ is the dual basis with respect to the intersection pairing; see (6) in [Mc2]. It follows from this identity and (63) that

$$\Psi_{A_{11},0,3}(pt; pt, \alpha_1, e_l) \neq 0$$

for some l. Taking $B = A_{11}$ we get (61). ∎

Proposition 7.4: *Let (M, ω) and (N, σ) be two closed symplectic manifolds. Then for every integer $k \geq 3$ and homology classes $A_2 \in H_2(N; \mathbb{Z})$ and $\beta_i \in H_*(N; \mathbb{Q})$, $i = 1, \ldots, k$,*

$$\Psi^{M\times N}_{0\oplus A_2,0,k}(pt; [M] \otimes \beta_1, \cdots, [M] \otimes \beta_{k-1}, pt \otimes \beta_k) = \Psi^N_{A_2,0,k}(pt; \beta_1, \ldots, \beta_k),$$

where $0 \in H_2(M; \mathbb{Z})$ denotes the zero class.

Proof: Take $J_M \in \mathcal{J}(M, \omega)$, $J_N \in \mathcal{J}(N, \sigma)$ and set $J = J_M \times J_N$. Note that the product symplectic manifold $(M \times N, \omega \oplus \sigma)$ is a special symplectic fibre bundle over (M, ω) with fibres (N, σ). Moreover, the almost complex structure $J = J_M \times J_N$ on $M \times N$ is **fibred** in the sense of Definition 2.2 in [Mc2]. So for a fibre class $0 \oplus A_2$ we can, as in the proof of Proposition 4.4 of [Mc2], construct a virtual moduli cycle $\overline{M}^{\nu}_{0,3}(M \times N, J, 0 \oplus A_2)$ of $\overline{M}_{0,3}(M \times N, J, 0 \oplus A_2)$ such that the M-components of each element in $\overline{M}^{\nu}_{0,3}(M \times N, J, 0 \oplus A_2)$ are

J_M-holomorphic, and thus constant. This shows that the virtual moduli cycle $\overline{M}^{\nu}_{0,3}(M \times N, J, 0 \oplus A_2)$ may be chosen as $M \times \overline{M}^{\nu}_{0,3}(N, J_N, A_2)$. The desired conclusion follows. These techniques were also used in the proof of Lemma 2.3 in [Lu6]. We refer to there and §4.3 in [Mc2] for more details. ∎

As a direct consequence of Proposition 7.3 and Proposition 7.4 we get

PROPOSITION 7.5: *The product of a closed symplectic manifold and a strong symplectic uniruled manifold is strong* 0*-symplectic uniruled. In particular, the product of finitely many strong symplectic uniruled manifolds is also strong* 0*-symplectic uniruled.*

Actually, we can generalize Proposition 7.4 to a symplectic fibre bundle over a closed symplectic manifold with a closed symplectic manifold as fibre. Therefore, a symplectic fibre bundle over a closed symplectic manifold with a strong symplectic uniruled fibre is also strong symplectic uniruled.

In the proof of Theorem 1.21 we need a product formula for Gromov–Witten invariants. Such a formula was given for algebraic geometry GW-invariants of two projective algebraic manifolds in [B]. However, it is not clear whether the GW-invariants used in this paper agree with those of [B] for projective algebraic manifolds. For the sake of simplicity we shall give a product formula for a special case, which is sufficient for the proof of Theorem 1.21. Recall that a symplectic manifold (M, ω) is said to be **monotone** if there exists a number $\lambda > 0$ such that $\omega(A) = \lambda c_1(A)$ for $A \in \pi_2(M)$. The **minimal Chern number** $N \geq 0$ of a symplectic manifold (M, ω) is defined by $\langle c_1, \pi_2(M)\rangle = N\mathbb{Z}$. For $J \in \mathcal{J}(M, \omega)$, a homology class $A \in H_2(M, \mathbb{Z})$ is called **J-effective** if it can be represented by a J-holomorphic sphere u: $\mathbb{C}P^1 \to M$. Such a homology class must be spherical. Moreover, a class $A \in H_2(M, \mathbb{Z})$ is called **indecomposable** if it cannot be decomposed as a sum $A = A_1 + \cdots + A_k$ of classes which are spherical and satisfy $\omega(A_i) > 0$ for $i = 1, \ldots, k$.

PROPOSITION 7.6: *Let the closed symplectic manifold (M, ω) either be monotone or have minimal Chern number $N \geq 2$. Then for each indecomposable class $A \in H_2(M, \mathbb{Z})$ and classes $\alpha_i \in H_*(M, \mathbb{Z})$, $i = 1, 2, 3$ the Gromov–Witten invariant $\Psi^M_{A,0,3}(pt; \alpha_1, \alpha_2, \alpha_3)$ adopted in this paper agrees with the invariant $\Psi^M_{A,3}(\alpha_1, \alpha_2, \alpha_3)$ in §7.4 of [McSa2].*

Proof: Let $J \in \mathcal{J}(M, \omega)$. Consider the space $\overline{\mathcal{M}}_{0,3}(M, A, J)$ of equivalence classes of all 3-pointed stable J-maps of genus zero and of class A in M. For $[\mathbf{f}] \in \overline{\mathcal{M}}_{0,3}(M, A, J)$, since A is indecomposable it follows from the definition of stable maps that $\mathbf{f} = (\Sigma; z_1, z_2, z_3; f)$ must be one of the following four cases:

(a) The domain $\Sigma = \mathbb{C}P^1$, z_i, $i = 1,2,3$ are three distinct marked points on Σ, and $f\colon \Sigma \to M$ is a J-holomorphic map of class A.

(b) The domain Σ has exactly two components $\Sigma_1 = \mathbb{C}P^1$ and $\Sigma_2 = \mathbb{C}P^1$ which have a unique intersection point. $f|_{\Sigma_1}$ is nonconstant and Σ_1 only contains one marked point. $f|_{\Sigma_2}$ is constant and Σ_2 contains two marked points.

(c) The domain Σ has exactly two components $\Sigma_1 = \mathbb{C}P^1$ and $\Sigma_2 = \mathbb{C}P^1$ which have a unique intersection point. $f|_{\Sigma_1}$ is nonconstant and Σ_1 contains no marked point. $f|_{\Sigma_2}$ is constant and Σ_2 contains three marked points.

(d) The domain Σ has exactly three components $\Sigma_1 = \mathbb{C}P^1$, $\Sigma_2 = \mathbb{C}P^1$ and $\Sigma_3 = \mathbb{C}P^1$. Σ_1 and Σ_2 (resp. Σ_2 and Σ_3) have only one intersection point, and Σ_1 and Σ_3 have no intersection point. $f|_{\Sigma_1}$ is nonconstant and Σ_1 contains no marked point. $f|_{\Sigma_2}$ is constant and Σ_2 contains one marked point. $f|_{\Sigma_3}$ is constant and Σ_3 contains two marked points.

Let $\overline{\mathcal{M}}_{0,3}(M,A,J)_i$, $i = 1,2,3,4$ be the subsets of the four kinds of stable maps. It is easily proved that for generic $J \in \mathcal{J}(M,\omega)$ they are smooth manifolds of dimensions

$$\begin{aligned}
\dim \overline{\mathcal{M}}_{0,3}(M,A,J)_1 &= \dim M + 2c_1(A),\\
\dim \overline{\mathcal{M}}_{0,3}(M,A,J)_2 &= \dim M + 2c_1(A) - 4,\\
\dim \overline{\mathcal{M}}_{0,3}(M,A,J)_3 &= \dim M + 2c_1(A) - 6,\\
\dim \overline{\mathcal{M}}_{0,3}(M,A,J)_4 &= \dim M + 2c_1(A) - 6.
\end{aligned}$$

So $\overline{\mathcal{M}}_{0,3}(M,A,J) = \bigcup_{i=1}^4 \overline{\mathcal{M}}_{0,3}(M,A,J)_i$ is a stratified smooth compact manifold. Note that each stable map in $\overline{\mathcal{M}}_{0,3}(M,A,J)$ has no free components. The construction of the virtual moduli cycle in [Lu8] with Liu–Tian's method in [LiuT] is thus trivial or not needed: The virtual moduli cycle of $\overline{\mathcal{M}}_{0,3}(M,A,J)$ may be taken as

$$\overline{\mathcal{M}}_{0,3}(M,A,J) \to \mathcal{B}^M_{0,3,A}, \quad [\mathbf{f}] \mapsto [\mathbf{f}],$$

where $\mathcal{B}^M_{0,3,A}$ is the space of equivalence classes of all 3-pointed stable $L^{k,p}$-maps of genus zero and of class A in M. Therefore for homology classes $\alpha_i \in H_2(M,\mathbb{Z})$, $i = 1,2,3$, satisfying the dimension condition

$$\deg(\alpha_1) + \deg(\alpha_2) + \deg(\alpha_3) = 2n + 2c_1(A)$$

the Gromov–Witten invariant

$$\begin{aligned}
\Psi^M_{A,0,3}(pt;\alpha_1,\alpha_2,\alpha_3) &= (\mathrm{EV}^{J,A}_{0,3})\cdot(\overline{\alpha}_1 \times \overline{\alpha}_2 \times \overline{\alpha}_3)\\
&= (\mathrm{EV}^{J,A}_{0,3})|_{\overline{\mathcal{M}}_{0,3}(M,A,J)_1}\cdot(\overline{\alpha}_1 \times \overline{\alpha}_2 \times \overline{\alpha}_3)
\end{aligned} \tag{64}$$

because the intersections can only occur in the top strata. Here

$$\text{(65)} \qquad \mathrm{EV}_{0,3}^{J,A} : \overline{\mathcal{M}}_{0,3}(M,A,J) \to M^3, \quad [\mathbf{f}] \mapsto (f(z_1), f(z_2), f(z_3)),$$

and $\overline{\alpha}_i : U_i \to M$ are generic pseudocycle representatives of the classes α_i, $i = 1,2,3$; cf. [McSa2] for details. Note that each element $[\mathbf{f}]$ in $\overline{\mathcal{M}}_{0,3}(M,A,J)_1$ has a unique representative of the form $(\mathbb{C}P^1; 0, 1, \infty; f)$. So $\overline{\mathcal{M}}_{0,3}(M,A,J)_1$ may be identified with the space $\mathcal{M}(M,A,J)$ of all J-holomorphic curves which represent the class A. Fix marked points $\mathbf{z} = (0,1,\infty) \in (\mathbb{C}P^1)^3$ and define the evaluation map

$$\text{(66)} \qquad E_{A,J,\mathbf{z}}: \mathcal{M}(M,A,J) \to M^3, \quad f \mapsto (f(0), f(1), f(\infty)).$$

From the above arguments one easily checks that it is a pseudocycle in the sense of [McSa2]. Then (64) gives rise to

$$\text{(67)} \qquad \Psi^M_{A,0,3}(pt; \alpha_1, \alpha_2, \alpha_3) = E_{A,J,\mathbf{z}} \cdot (\overline{\alpha}_1 \times \overline{\alpha}_2 \times \overline{\alpha}_3) = \Psi^M_{A,3}(\alpha_1, \alpha_2, \alpha_3)$$

because we can require that $\overline{\alpha}_1 \times \overline{\alpha}_2 \times \overline{\alpha}_3$ is also transverse to $E_{A,J,\mathbf{z}}$. ∎

PROPOSITION 7.7: *Consider closed symplectic manifolds (M_k, ω_k) as in Proposition 7.6 and indecomposable classes $A_k \in H_2(M_k, \mathbb{Z})$, $k = 1, \dots, m$. Then for $\alpha_i^{(k)} \in H_*(M_k, \mathbb{Z})$, $i = 1,2,3$ and $k = 1, \dots, m$ we have the Gromov–Witten invariant*

(68)

$$\Psi^M_{A,0,3}(pt; \times_{k=1}^m \alpha_1^{(k)}, \times_{k=1}^m \alpha_2^{(k)}, \times_{k=1}^m \alpha_3^{(k)}) = \prod_{k=1}^m \Psi^{M_k}_{A_k,0,3}(pt; \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}),$$

where $A = \bigoplus_{k=1}^m A_k$.

Proof: Set $(M, \omega) = (\times_{k=1}^m M_k, \times_{k=1}^m \omega_k)$. Take $J_k \in \mathcal{J}(M_k, \omega_k)$, $k = 1, \dots, m$ and set $J = \times_{k=1}^m J_k$. Then $J \in \mathcal{J}(M, \omega)$. It is not hard to prove that for generic $J_k \in \mathcal{J}(M_k, \omega_k)$ the space $\overline{\mathcal{M}}_{0,3}(M,A,J)$ is still a stratified smooth compact manifold. We still denote by $\overline{\mathcal{M}}_{0,3}(M,A,J)_1$ its top stratum, which consists of elements $[\mathbf{f}] \in \overline{\mathcal{M}}_{0,3}(M,A,J)$ whose domain has only one component $\mathbb{C}P^1$. It is a smooth noncompact manifold of dimension $\dim M + 2c_1(A) = \sum_{k=1}^m \dim M_k + 2c_1(A_k)$, and each element $[\mathbf{f}] \in \overline{\mathcal{M}}_{0,3}(M,A,J)_1$ has a unique representative of the form

$$\mathbf{f} = (\mathbb{C}P^1; 0, 1, \infty; f = (f_1, \dots, f_m)),$$

where $f_k : \mathbb{C}P^1 \to M_k$ are J-holomorphic maps in the homology classes A_k, $k = 1, \dots, m$. Note that the other strata of $\overline{\mathcal{M}}_{0,3}(M, A, J)$ have at least codimension two. For homology classes $\alpha_i^{(k)} \in H_*(M_k, \mathbb{Z})$, $i = 1,2,3$ and $k = 1, \dots, m$, satisfying the dimension condition

$$\deg(\alpha_1^{(k)}) + \deg(\alpha_2^{(k)}) + \deg(\alpha_3^{(k)}) = \dim M_k + 2c_1(A_k),$$

we may choose the pseudo-cycle representatives $\overline{\alpha}_i^{(k)}: U_i^{(k)} \to M$, $i = 1,2,3$ and $k = 1, \dots, m$ such that:

(i) $(\times_{k=1}^m \overline{\alpha}_1^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_2^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_3^{(k)})$ is transverse to the evaluations $\mathrm{EV}_{0,3}^{J,A}$ in (65) and $E_{A,J,\mathbf{z}}$ in (66),

(ii) each $\overline{\alpha}_1^{(k)} \times \overline{\alpha}_2^{(k)} \times \overline{\alpha}_3^{(k)}$ is transverse to the evaluations $E_{A_k,J_k,\mathbf{z}}$ and

$$\mathrm{EV}_{0,3}^{J_k,A_k} : \overline{\mathcal{M}}_{0,3}(M_k, A_k, J_k) \to M_k^3, \quad [\mathbf{f}_k] \mapsto (f_k(0), f_k(1), f_k(\infty))$$

for $k = 1, \dots, m$.

Then as above we get that the Gromov–Witten invariant

$$\begin{aligned}
&\Psi_{A,0,3}^M(pt; \times_{k=1}^m \alpha_1^{(k)}, \times_{k=1}^m \alpha_2^{(k)}, \times_{k=1}^m \alpha_3^{(k)}) \\
&= (\mathrm{EV}_{0,3}^{J,A}) \cdot ((\times_{k=1}^m \overline{\alpha}_1^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_2^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_3^{(k)})) \\
&= (\mathrm{EV}_{0,3}^{J,A})|_{\overline{\mathcal{M}}_{0,3}(M,A,J)_1} \cdot ((\times_{k=1}^m \overline{\alpha}_1^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_2^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_3^{(k)})) \\
&= E_{A,J,\mathbf{z}} \cdot ((\times_{k=1}^m \overline{\alpha}_1^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_2^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_3^{(k)}))
\end{aligned} \tag{69}$$

because of (67). Note that $\mathcal{M}(M, A, J) = \prod_{k=1}^m \mathcal{M}(M_k, A_k, J_k)$. It easily follows from the above (i) and (ii) that

$$\begin{aligned}
E_{A,J,\mathbf{z}} \cdot ((\times_{k=1}^m \overline{\alpha}_1^{(k)}) &\times (\times_{k=1}^m \overline{\alpha}_2^{(k)}) \times (\times_{k=1}^m \overline{\alpha}_3^{(k)})) \\
&= \prod_{k=1}^m E_{A_k,J_k,\mathbf{z}} \cdot (\overline{\alpha}_1^{(k)} \times \overline{\alpha}_2^{(k)} \times \overline{\alpha}_3^{(k)}) \\
&= \prod_{k=1}^m \Psi_{A_k,3}^{M_k}(\alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}) \\
&= \prod_{k=1}^m \Psi_{A_k,0,3}^{M_k}(pt; \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}).
\end{aligned}$$

The final step comes from Proposition 7.6. This and (69) lead to (68). ∎

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